Directed Graphs and Rectangular Layouts (Extended abstract)

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Abstract. This paper deals with the problem, arising in practice, of drawing a directed graph as a collection of disjoint, isothetic rectangles, where the rectangles of the nodes of each edge must touch and where the placement of the rectangles respects the ordering of the edges. It provides characterizations for those graphs having the special type of rectangular layout known as a rectangular dual. It then characterizes the st-graphs having rectangular layouts in terms of the existence of certain planar embeddings and the non-existence of a particular subgraph.

1 Introduction

We consider the problem of drawing a directed graph using disjoint, isothetic rectangles for the nodes, such that (u, v) is an edge in the graph if and only if the corresponding rectangles R_u and R_v touch, with R_u above or to the left of R_v. Figure 1 provides an example, showing a directed tree drawn both using the traditional, straight-line drawing and as a collection of rectangles.

The problem arose in the context of providing a graphical user interface to a relational database support system which allows users to model and administer their database. Users can specify their own schemas using entity-relationship models [17]. It is assumed that there are no cycles among the relationships in the schema. The system then displays the database entities as rectangular buttons. Clicking on a button provides information related to the corresponding entity type, such as detailed descriptions of the entity attributes or information about specific records. Experience indicates the benefit of juxtaposing buttons that correspond to related entities. In addition, if the relationship is one-to-many, the button corresponding to the "many" entity is positioned below or to the right of the "one" entity, to emphasize this sense of directionality. There are no additional constraints. In particular, even if two entities are not related, their rectangles may abut. Viewing the database schema as the obvious directed tree, with entities as vertices and relations as edges, and allowing undirected cycles, leads to the graph drawing problem stated above.[1]

This generalizes the concept of *rectangular layouts*, in which an undirected graph G is drawn using disjoint rectangles for nodes, and two nodes are adjacent in G if and only if the two corresponding rectangles are adjacent. Without directionality, this [10, 11,

[1] The implied additional restriction that rectangles can only touch if there is an edge is unimportant (cf. Section 2).

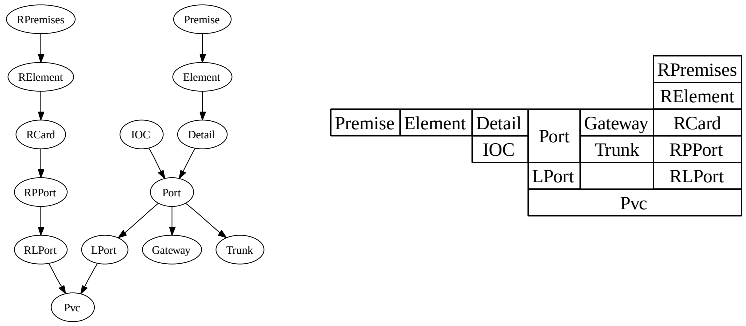


Fig. 1. Traditional drawing and rectangular layout of tree.

3, 12, 6, 9, 14, 7, 15, 16, 5] and related problems, such as proximity drawing [2, 8] and rectangle of influence drawings [13, 4], have been extensively studied. In particular, there are effective characterizations for graphs having rectangular layouts, and some provable bounds on the size of the layouts.

In this paper, we explore what effect the directionality constraint has. In Section 2, we formally define the problem and note some basic results. In Section 3, we consider the special case of rectangular duals, where the rectangles in a rectangular layout form one large rectangle. We give several characterizations for graphs having a rectangular dual. Then, in Section 4, we show that, for st-graphs, we have the same characterization as in the undirected case after accounting for one forbidden subgraph.

2 Preliminaries

Throughout, we assume that all graphs are directed and connected, with no loops or multiple edges, and have at least 4 vertices.

A *(strong) rectangular layout* of a graph $G = (V, E)$ is a set R of isothetic rectangles whose interiors are pairwise disjoint, with an isomorphism $\mathcal{R} : V \rightarrow R$ such that for any two vertices $u, v \in V$, the boundaries of $R_u = \mathcal{R}(u)$ and $R_v = \mathcal{R}(v)$ overlap non-trivially with

$$\mathcal{R}(u) \text{ above or to the left of } \mathcal{R}(v) \tag{1}$$

if and only if $(u, v) \in E$. Figure 2 gives two graphs and sample rectangular layouts.

It is immediate that only acyclic graphs can have a rectangular layout. Thus, we assume that any graph is a directed acyclic graph (*dag*).

A *weak rectangular layout* of G is a rectangular layout in which two nodes might touch even if there is no edge between the corresponding nodes. Note that, in Figure 1, layout (b) is a weak layout for both graphs (a) and (b) but a strong layout only for graph (b). The reader may have also noticed that the rectangular layout in Figure 1 is weak, with the rectangles for nodes Detail and IOC touching but with no edge connecting them.

For purposes of construction, however, the distinction between strong and weak layouts does not matter, as shown by the following lemma.

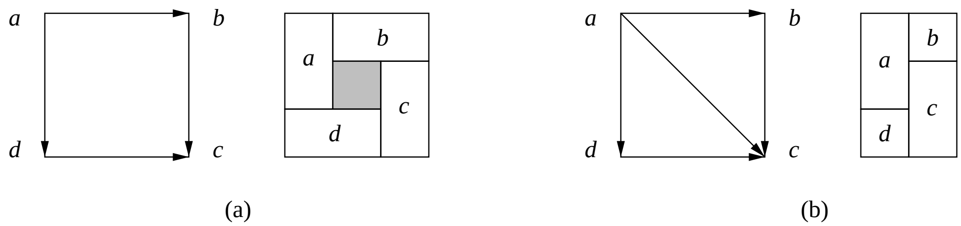


Fig. 2. Two graphs and associated rectangular layouts. The shaded region depicts a gap.

Lemma 1. *Graph G has a weak rectangular layout R if and only if G has some strong rectangular layout L.*

Proof. The shearing argument given in Buchsbaum et al.[5] extends to the directed case.

Furthermore, the assumption that no two rectangles meet trivially at a corner can be relaxed. If rectangles R_u and R_v meet at a corner, we can perturb the boundaries by some small amount to make the boundary overlap non-trivial. The layout becomes weak if it was not already. Lemma 1 shows that it can be made strong with only non-trivial boundary overlaps.

3 Directed rectangular duals

For the case of rectangular layouts of undirected graphs, it has been useful (e.g., see [5, 6, 10–12]) to consider the case where the rectangles form a partition of an enclosing rectangle. We take a similar approach here. A *rectangular dual* of a graph G is a rectangular layout in which the union of the rectangles is a rectangle.

Let G be a planar st-graph[2] with embedding E. Let $b_r = (v_0 = s, v_2, ..., v_n = t)$ be the right boundary path of E from s to t. Let $b_l = (u_0 = s, u_2, ..., u_m = t)$ be the left boundary path of E from s to t. Any vertex not in b_r or b_l is an *interior* vertex. Add four vertices $N, E, S, and\ W$ so that N is above and to the right of G, W is above and to the left of G, S is below and to the left of G, and E is below and to the right of G. We say the boundary of E is *embeddable* if we can find $0 \leq i \leq n$ and $0 \leq j \leq m$ such that G extended with the nodes N, E, S, and W and the edges $(N, v_k), 0 \leq k \leq i$, $(v_k, E), i \leq k \leq n$, $(W, u_k), 0 \leq k \leq j$, and $(u_k, S), j \leq k \leq m$, is planar, and all of the vertices v_k and u_k have indegree and outdegree ≥ 2.

Theorem 1. *A dag G has a directed rectangular dual if and only if G is a planar st-graph with an embedding E such that:*

1. *all interior faces are triangles*
2. *all interior vertices have indegree and outdegree ≥ 2.*

[2] Concerning the definition and basic properties of st-graphs, see [1]. For all embeddings of st-graphs, we assume that s is above t and all edges are directed downward.

3. *the boundary of E is embeddable.*

Proof. For necessity, the first two conditions are obvious. As for the third, if G has a rectangular dual, attaching 4 rectangles to the sides of the layout shows that the boundary is embeddable.

Let G' be G extended with the vertices N, E, S, W, the edges from condition 3), and the edges (N, E) and (W, S). Note that for any vertex v in G, its rightmost inedge and outedge form two edges of a triangle in G'. The similar property holds for the leftmost inedge and outedge.

For sufficiency, we show how to construct a regular edge labeling (*REL*) in the sense of He [6] on G'. Given an REL, He's construction produces a rectangular dual for the underlying undirected graph of G'. It is simple to verify that the construction honors edge directions, thereby giving us a (directed) rectangular dual for G'.

We now show how to construct the necessary REL. Recall that an REL is a partition of the directed edges[3] into 2 sets T_1 and T_2 such that, for each vertex v, the edges adjacent to v consist of the inedges of T_1, followed by the inedges of T_2, followed by the outedges of T_1, followed by the outedges of T_2, in the counterclockwise direction. Each of the four subsets must be non-empty.

We define T_1 to include the leftmost outedge and the rightmost inedge of every vertex. Let T_2 be the complement of T_1. To show that the four subsets of edges around a vertex are non-empty, it suffices to show that the leftmost inedge and the rightmost outedge of each vertex will be in T_2. Let $e = (v, w)$ be the rightmost outedge of v. Since v has outdegree at least 2, e cannot be the leftmost outedge. In addition, if (u, v) is the rightmost inedge of v, we must have a face $\{u, v, w\}$ with an edge (u, w), so e cannot be the rightmost inedge of w. Thus, e must be in T_2. The symmetric argument holds for the leftmost inedge of v.

To see that the edges are appropriately ordered, let $e = (v, w)$ be an outedge of v in T_1. If e is not the leftmost outedge, let $e' = (v, u)$ be the next outedge to the left. Since e is in T_1 and e is not the leftmost outedge of v, we must have that e is the rightmost inedge of w. Since w has indegree at least 2, there must be another inedge of w just to the left of e. This means that $\{v, u, w\}$ forms a face and e' is the rightmost inedge of u, so that e' is in T_1. This implies that any outedge to the left of an outedge in T_1 is also in T_1, and therefor all T_1 are to the left of all T_2 outedges. A symmetric argument holds for the inedges.

Figure 3(a) exhibits an embedded planar st-graph. Figure 3(b) shows the edges belonging to the sets T_1 and T_2 as defined in the theorem.

Once the four vertices N, E, S, W are attached, the amount of variation in constructing an REL is limited by the following theorem. We say an edge is a *middle* edge if has the form shown by the dotted edge in Figure 4.

Theorem 2. *If G is a dag satisfying the three conditions of Theorem 1, then any edge e satisfies exactly one of the following:*

1. *e is the leftmost inedge and/or the rightmost outedge of a vertex*

[3] For undirected graphs, constructing an REL requires that a direction be assigned to each edge. Here we use the given direction.

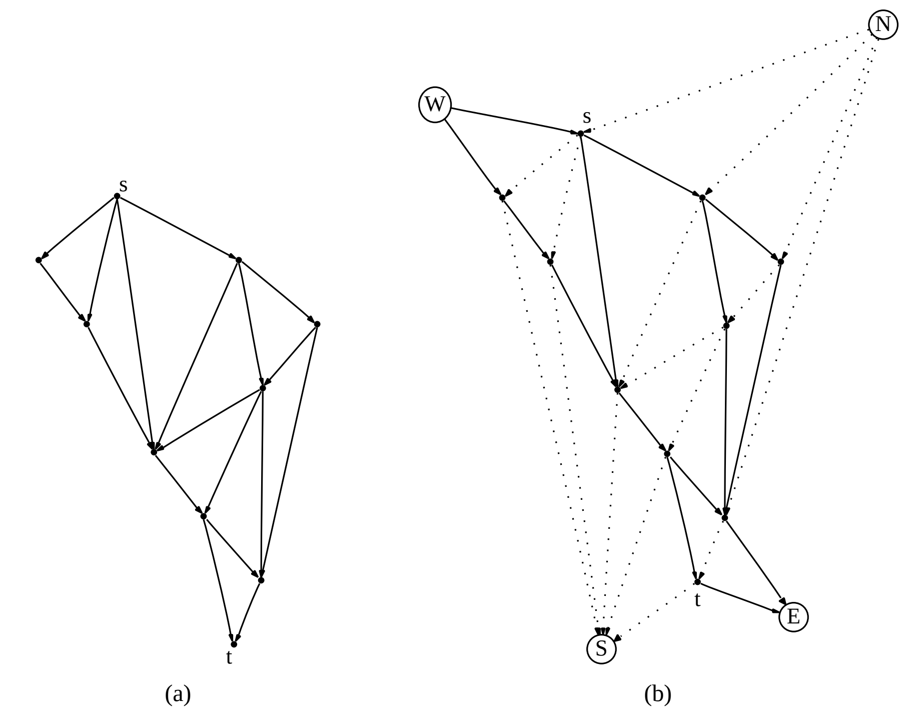


Fig. 3. (a) st-graph G (b) extended st-graph G with edge sets T_1 (dotted) and T_2

2. *e is the leftmost outedge and/or the rightmost inedge of a vertex*
3. *e is a middle edge*

Proof. We have already observed, as part of the proof of Theorem 1, that the first two sets are disjoint. To complete the proof, we simply have to show that any edge not in one of the first two sets must be a middle edge. This is almost immediate. If $e = (u, v)$ is not the rightmost outedge of u nor the rightmost inedge of v, the face to the right of e must have be composed of the three edges $(u, v), (u, w), (w, v)$ for some vertex w. Since the same must be true for face to the left of e, we see that e is a middle edge.

Theorem 2 shows that the only choices in constructing an REL, once the graph is embedded, concern where to put the middle edges. It is a simple observation that any middle edge can be in either T_1 or T_2, so the number of distinct directed rectangular duals is 2^m, where m is the number of middle edges.

There is a simple sufficient condition for property 3) in Theorem 1, namely, that, on the right and left boundaries of E, no vertex with indegree 1 appears after any vertex with outdegree 1. Consider the right boundary $b_r = (v_0 = s, v_2, ..., v_n = t)$ of E. Let k be the largest index so that v_k has indegree < 2. Let l be the smallest index so that v_l has outdegree < 2. Let i be any value $k \leq i \leq l$, and add the edges $(N, v_k), 0 \leq k \leq i$ and $(v_k, E), i \leq k \leq n$. By hypothesis, the extended graph is planar. Each v_k in G has indegree and outdegree at least 1. If it has indegree 1, the edge (N, v_k) was added, so it

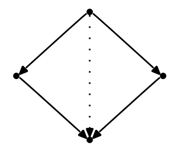


Fig. 4. Middle edge

has indegree 2 in the extended graph. The analogous results hold for the outdegrees in b_r, and for the left boundary of E.

This condition characterizes the biconnected graphs with directed rectangular duals. (We consider a dag biconnected or not according to that property holding in the underlying undirected graph.)

Theorem 3. *A biconnected dag G has a directed rectangular dual if and only if G is a planar st-graph with an embedding E such that:*

1. *all interior faces are triangles*
2. *all interior vertices have indegree and outdegree ≥ 2.*
3. *on the right and left boundaries of E, no vertex with indegree 1 appears after any vertices with outdegree 1.*

Proof. We have already noted how the three properties, along with Theorem 1, imply the existence of a directed rectangular dual. Conversely, if G has a directed rectangular dual, any vertex v on b_r with indegree 1 must correspond to a rectangle on the top boundary. Any vertex u on b_r appearing before v is therefore also on the top boundary with a rectangle to its right. Since G is biconnected, u cannot also lie along the bottom boundary, as then it would be an articulation point. Thus, u has a rectangle below it. This shows it has outdegree ≥ 2.

It is simple to apply this result to the cases where the condition 3) is trivially satisfied.

Corollary 1. *A biconnected dag G has a directed rectangular dual with a single rectangle on both the right and left sides if and only if it satisfies properties 1) and 2) of Theorem 3 and all vertices of one boundary path have outdegree ≥ 2, and all vertices of the other boundary path have indegree ≥ 2.*

Analogous results hold when the degree constraints are only applied to one side.

Theorem 3 clearly fails for non-biconnected graphs. A chain has a rectangular dual but condition 3) does not hold. The next theorem, however, allows us to apply the theorem to the block decomposition of non-biconnected graph.

Theorem 4. *A dag G has a rectangular dual if and only if*

1. *its block tree is a chain*
2. *each interior block has a directed rectangular dual with a single rectangle on both the left and right sides*

3. *both the first and last blocks have rectangular duals, with a single rectangle on the right and left side, respectively.*

Proof. Given the 3 conditions, joining the rectangular duals in the order specified by the block tree chain, abutting or identifying the side rectangles as necessary, gives a dual for the whole graph. On the other hand, if G has a rectangular dual, any rectangle R_v corresponding to a cut point v must be as wide or as high as the whole rectangle. By a $90°$ rotation, we can assume it is as high. By dividing the rectangular dual at R_v, splitting R_v into 2 rectangles if both its indegree and outdegree are greater than 1, we end up with rectangular duals for 2 proper subgraphs of G, and can complete the proof by induction using Theorem 3 for the base case.

4 st-Graphs with rectangular layouts

As with the undirected case, any graph with a (directed) rectangular layout is a subgraph of some graph with a (directed) rectangular dual. Thus, the results from Section 3 can help us characterize certain classes of graphs with rectangular layouts.

We start with some definitions and lemmas. We let $\mathcal{F}$ denote the graph shown in Figure 5(a). From the next lemma, we also refer to $\mathcal{F}$ as *forbidden*.

Lemma 2. *Let G be a biconnected planar st-graph with an embedding such that all interior faces are triangles. The condition that all interior vertices have indegree and outdegree ≥ 2 is equivalent to G not containing a subgraph isomorphic to $\mathcal{F}$.*

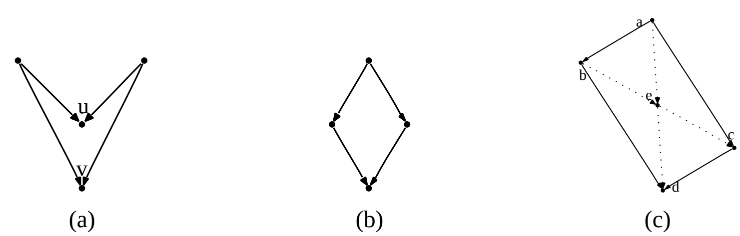


Fig. 5. (a) The forbidden graph $\mathcal{F}$. (b) Quadrilateral face (c) Triangulation of a quadrilateral face

Proof. If G has $\mathcal{F}$ as a subgraph, we can assume, by symmetry, that its embedding is as shown in Figure 5(a). Consider the set formed by u and all vertices within the quadrilateral, and pick one w with the minimum y coordinate. Then the only possible outedge is (w, v), and w has outdegree 1.

To show the opposite direction, we can assume, without loss of generality, that G has a vertex u with outdegree 1. Let (u, v) be the unique outedge. The two triangular faces common to (u, v) must have the form $(w, u), (w, v), (u, v)$ and $(z, u), (z, v), (u, v)$. Therefore, G has the subgraph $(w, u), (w, v), (z, u), (z, v)$ isomorphic to $\mathcal{F}$.

For a planar embedding of a graph, a *filled triangle* is defined to be a length-3 cycle with at least one vertex inside the induced region.

Lemma 3. *Let E be an embedding of a planar st-graph G. If E has a filled triangle, G has an interior vertex with outdegree or indegree 1.*

Proof. Consider a filled triangle with edges (v, u), (v, w) and (u, w). If there is any vertex inside the triangle whose y coordinate is the same or less than that of u, pick the one x with the smallest y coordinate. Then the only possible outedge of x is (x, w) and x has outdegree 1. A symmetric argument holds if any vertex inside the triangle has y coordinate the same or more than that of u.

We can now characterize planar st-graphs having a rectangular layout.

Theorem 5. *Let G be a planar st-graph. G has a rectangular layout if and only if G has no subgraph isomorphic to $\mathcal{F}$ and has an embedding with no filled triangles.*

Proof. $\mathcal{F}$ has no rectangular layout, so it cannot be a subgraph of any graph having one. In addition, if G has a rectangular layout, it is a subgraph of a graph H with a rectangular dual. By Theorem 1, H has an embedding with all interior faces triangles and all interior vertices having outdegree and indegree greater than 1. By Lemma 3, the embedding for H, and hence G, has no filled triangle.

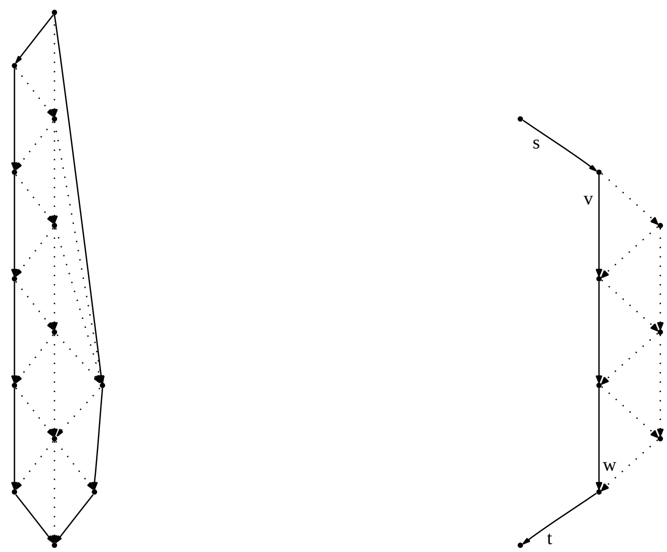
To show the converse, first assume that G is biconnected and let E be an embedding of G. For each non-triangular face F, let n be the length of the longest boundary path (i.e., the path has $n + 1$ vertices). If $n = 2$, then F must look like Figure 5(b). Add a single vertex and four edges to form the graph in Figure 5(c). The added edges are dotted. To maintain a valid planar embedding of an st-graph, we need to vertically shift down node c and all nodes reachable from it by a directed path by a uniform amount, as indicated in the figure.

If $n > 2$, insert $n - 2$ vertices to form a path in the interior from the top vertex of F to the bottom. Alternately connect the vertices along the longest boundary path to the inserted vertices as indicated in Figure 6(a). The added edges are dotted. Similarly triangulate the shorter side, connecting the highest boundary path node to all of the remaining added vertices. None of these operations can introduce a filled triangle, a cut vertex, or a subgraph isomorphic to $\mathcal{F}$.

After processing all non-triangular faces, the resulting graph is still biconnected and acyclic, with a planar embedding with all interior faces triangular. In addition, there is no subgraph isomorphic to $\mathcal{F}$. Thus, by Lemma 2, every interior vertex has indegree and outdegree at least 2.

To be able to invoke Theorem 3, we need to make sure the graph satisfies property 3. For a given boundary path, if there is any node of indegree 1 appearing after a node of outdegree 1, let n be the number of vertices on the boundary path from s to t. The cases where $n < 4$ are trivial, so we can assume $n \geq 4$. Let the vertices on the boundary path be $s, v, \ldots, w, t$. Add $n - 3$ new vertices on a path from v to w. Then triangulate the face as indicated in Figure 6(b) for the case $n = 6$. As usual, the added edges are dotted. This process eliminates any vertices of outdegree 1 before a vertex of indegree 1, and maintains the hypotheses of the theorem.

We now can apply Theorem 3 to get a rectangular dual for the extended graph. Removing the rectangles corresponding to added vertices gives a rectangular layout for G.



(a) (b)

Fig. 6. (a) Triangulation of a non-quadrilateral face (b) Triangulating a boundary path

Finally, if G is not biconnected, we can use induction, with the base case of a biconnected graph dealt with above. Pick a cut vertex v in G. Possibly replicating v if both its indegree and outdegree are greater than 1, we can split G at v into two smaller st-graphs, each satisfying the hypotheses of the theorem. By induction, we get rectangular layouts for each. Necessarily, in any rectangular layout of an st-graph, the source and sink nodes correspond to the upper left and lower right rectangles, respectively. As in Theorem 4, we can join the layouts together, identifying rectangles if necessary for replicated nodes, to obtain a rectangular layout for G.

Except for the forbidden subgraph, Theorem 5 is identical to Corollary 3.5 in [5]. Note that if all face boundary paths have length at least 2, there can be no filled triangles. The example of Figure 7(a) shows how the triangulation used in the proof of the theorem fails in the case of filled triangles. The graph is the same that of Figure 5(c) but we assume it also has an edge (a, d), giving the filled triangle $(a, d), (a, c), (c, d)$. When the auxiliary node (e) and (dotted) edges are added, we have a forbidden subgraph $(a, d), (e, d), (a, c), (e, c)$.

Both conditions of the theorem are necessary in order to have a rectangular layout. Figure 7(b) shows an example of an st-graph with no forbidden subgraph all of whose embeddings have a filled triangle. Figure 7(c) shows an example of an st-graph with a forbidden subgraph but no filled triangle. Of course, neither has a rectangular layout.

We believe Theorem 5 can be extended to general dags as follows:

Conjecture 1. Let G be a dag. G has a directed rectangular layout if and only if G has no subgraph isomorphic to $\mathcal{F}$ and has an upward planar embedding with no filled triangles.

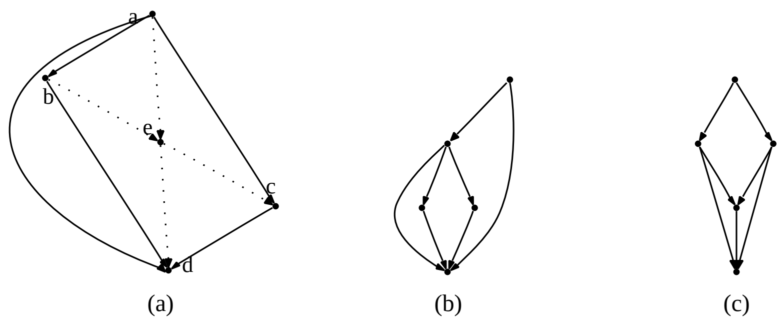


Fig. 7. (a) Filled triangle yielding $\mathcal{F}$ (b) All embeddings with filled triangle (c) Subgraph $\mathcal{F}$ but no filled triangle

The idea of the proof should be the same as that for Theorem 5 but the bookkeeping is more complex.

5 Summary and future work

If this paper, we considered the extension of the problem of rectangular layouts to directed graphs. We presented characterizations for which graphs have rectangular duals, and which st-graphs admit rectangular layouts.

As for future work, we have already noted the conjecture at the end of the previous section. In analogy with the undirected case, there are many avenues open for further research. In particular, given the original motivation for the problem, it would be nice to have a layout algorithm with provably tractable running time and provable area or width, bounds similar to the results presented by Buchsbaum et al. [5] for the undirected problem.

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