Low-Dimensional Lattices I:
Quadratic Forms of Small Determinant

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Abstract

In this paper, the first of a series dealing with low-dimensional lattices and their applications, we classify positive-definite integral quadratic forms of determinant $d \le 25$ in dimensions up to a limit which ranges from 18 (for $d = 1$) to 7 (for $d = 25$). [New version Nov. 5, 2000. A different version of this paper appeared in Proc. Royal. Soc. London, Series A, Vol. 418 (1988), pp. 17—41.]

1. Introduction

This paper is the first in a series dealing with low-dimensional lattices and their applications. Here the application is to number theory: we classify all positive-definite integral quadratic forms of sufficiently small determinant and dimension. In later papers we intend to discuss maximal subgroups of $GL(n, \mathbf{Z})$ (in Part II), perfect forms (in Part III), the mass formulae (in Part IV), and possibly other topics.

One motivation for writing this series of papers is to present a systematic notation for these lattices. The need for this is illustrated by the fact that the symbols

V_5 (Korkine & Zolotareff, 1877),

ϕ_1 in five dimensions (Voronoi, 1907),

B_5 or D_5 (Coxeter, 1951), (Barnes, 1959),

F_2 in five dimensions (Plesken & Pohst, 1977),

Λ_5 (Conway & Sloane, 1982a),

P_5^1 (Conway & Sloane, 1988b)

all name the same lattice. Of course some multiplicity of names is both appropriate and inevitable — this lattice is indeed the five-dimensional *laminated lattice* Λ_5 and is also the absolutely extreme five-dimensional lattice P_5^1. However, we feel that this lattice is usually best regarded as the root lattice D_5.

Thus one of our main goals is to simplify and systematize the work of others, rather than to present new material. Nevertheless we do include a number of new results.

In Part I, the present paper, the underlying ideas are largely due to Witt (1941) and Kneser (1957). We apply them to produce a table of quadratic forms that goes considerably further than any previous table.

In Part II we follow the work of Dade, Plesken & Pohst (1977), and others, who classified the maximal subgroups of $GL(n, \mathbf{Z})$ for various values of n. Plesken and Pohst describe their groups as automorphism groups of lattices specified merely by Gram matrices. We shall describe these lattices in a more invariant way, which makes the groups more visible.

In Part III we perform a similar service for the perfect lattices in up to seven dimensions found by Barnes, Scott, Larmouth, Stacey and others. We also provide a simple proof for the classification of perfect lattices in up to four dimensions.

Topics which may be discussed in later parts include the Minkowski-Siegel mass formulae, G. L. Watson's one-class genera of quadratic forms, and the completion of the enumeration of septenary perfect lattices.

In the present paper, the main table (Table 1) classifies all indecomposable positive-definite integral quadratic forms of given determinant $d \le 23$, in all dimensions below the *critical dimension* — the dimension in which indecomposable *congeners* first appear, i.e. inequivalent forms belonging to the same genus. Since in preparing this table we also classified the forms in the critical dimension, we describe such forms briefly in a supplementary table (Table 2). The reader will find that these tables, together with the examples given in §§4-6, actually classify all indecomposable forms of determinant at most 25 and dimension at most 7. Table 0 provides an overall summary.

Ultimately, of course, the number of distinct lattices of a given determinant is super-exponential in the dimension. We were therefore surprised to see how far the classification can be carried. For example it can be seen from Table 1 that there are just two indecomposable seven-dimensional forms of determinant 11, of shapes $E_6\ 33_1$ and $A_6\ 77_1$. Since these belong to different genera ($\mathrm{I}_7(11^+)$ and $\mathrm{I}_7(11^-)$), they can be simply called 11_7^+ and 11_7^-.

We have been unable to find any prior tables of comparable extent. Kneser (1957) enumerated lattices with determinants $d = 1, 2, 3$ for dimensions $n \le 17 - d$. Unimodular ($d = 1$) lattices have since been classified for dimensions $n \le 25$ in (Niemeier, 1973), (Conway & Sloane, 1982, 1982b), and (Borcherds, 1984, 1988), and lattices with $d = 2, 3$ for $n \le 20 - d$ in (Conway & Sloane, 1988, Chap. 15).

Sections 4-6 describe the enumeration techniques we used. Our basic strategy is to reduce the problem of enumerating the lattices of a given determinant $d \geq 1$ to a lower determinant, and thus ultimately to the case $d = 1$, where we can make use of the tables in (Conway & Sloane, 1988). We have used two methods, called ‘‘complementation’’ (illustrated in §4 by the enumeration of lattices with determinant $d = 23$) and ‘‘supplementation’’ (illustrated in §5 by the enumeration for $d = 24$). The techniques are further illustrated in §6 where both methods are needed to enumerate the lattices with $d = 25$. Standard classification methods using reduced forms would be totally impracticable in these dimensions.

Our techniques also make considerable use of properties of the *root lattices* A_n, D_n, E_6, E_7, E_8 (such as the structure of $E_6^* / 2E_6^*$, the transitivity of E_8 on sublattices of type A_r, etc). Section 2 contains a brief description of root lattices and gluing theory, and the genus of a lattice or of the associated quadratic form is discussed in §3. The tables will be found in §7, and the Appendix describes the glue vectors and related properties of the root lattices.

The results have all been checked by use of the mass formulae (see Part IV).

Notation

In real Euclidean space $\mathbf{R}^n$ equipped with inner product $(v, w) = v \cdot w$, a (positive-definite) *lattice* L consists of all integral linear combinations

$$v = \zeta_1 v_1 + \cdots + \zeta_n v_n \qquad (\zeta_i \in \mathbf{Z})$$

of n linearly independent vectors $v_1, \ldots, v_n$. The vectors $v_1, \ldots, v_n$ form an *integral base* for L, and

$$f(\zeta) = (v, v) = \zeta A \zeta^{tr}$$

with

$$\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{Z}^n ,$$
$$A = (a_{ij}) , \quad a_{ij} = (v_i, v_j) ,$$

is the corresponding quadratic form. We shall normally speak of lattices rather than quadratic forms, although it is to be understood that the same terminology applies to both concepts. The number $N(v) = (v, v)$ is called the *norm* of v, and the minimal value μ of (v, v) for $v \in L$, $v \neq 0$, is the *minimal norm* of L. Also A is a *Gram matrix* for L, and $d = \det A$ is the *determinant* of this lattice.

The *dual* lattice L^* is defined by

$$L^* = \{ u \in \mathbf{R}^n : (u, v) \in \mathbf{Z} \text{ for all } v \in L \} .$$

If $L \subseteq L^*$, L is called *integral*. In the present paper ‘‘lattice’’ will always mean ‘‘positive-definite integral lattice’’.

Two lattices L and M are *integrally equivalent*, written $L \cong M$, if one can be obtained from the other by a rotation and (possibly) a reflection. The subscript on the name for a lattice gives its dimension. For other undefined terms and further background information see (Cassels, 1978) or (Conway & Sloane, 1988).

2. Root lattices and gluing theory

The strength of the Witt-Kneser method (see (Witt, 1941), (Kneser, 1957)) arises from the fact that a lattice of sufficiently small determinant and dimension will contain a large sublattice generated by vectors of norms 1 and 2, and such sublattices are completely classified.

In particular, *Witt's theorem* states that, in any integral lattice L, the sublattice generated by vectors of norms 1 and 2 is a direct sum of a lattice I_m $(m \geq 0)$ and *root lattices* taken from the list A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7 and E_8. (We call this sublattice the Witt part of L — see the end of this section.)

These lattices are defined as follows. I_n $(n \geq 0)$ is the ordinary integer lattice consisting of all points $(x_1, \ldots, x_n)$, $x_i \in \mathbf{Z}$, in Euclidean space R^n equipped with the standard quadratic form $x_1^2 + \cdots + x_n^2$. The determinant of I_n is $d = 1$. A_n $(n \geq 1)$ is the sublattice of I_{n+1} satisfying

$$x_1 + \cdots + x_{n+1} = 0 ,$$

and has determinant $d = n + 1$. D_n $(n \geq 4)$ is the sublattice of I_n satisfying

$$x_1 + \cdots + x_n \text{ even} ,$$

and has determinant $d = 4$. E_8 is the lattice generated by D_8 and the vector $(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$. We may define E_7 to be the sublattice of E_8 satisfying

$$x_1 + \cdots + x_8 = 0 ,$$

and E_6 to be the sublattice of E_8 satisfying

$$x_1 + x_8 = x_2 + \cdots + x_7 = 0 .$$

E_8, E_7, E_6 have determinants $d = 1, 2, 3$ respectively.

To describe the structure of a lattice that contains a direct sum of I_m's, A_r's, D_s's and E_t's as a sublattice we use gluing theory. *Gluing theory* is a way to describe the general n-dimensional integral lattice L that has a sublattice which is a direct sum

$$L_1 \oplus L_2 \oplus \cdots \oplus L_k$$

of given integral lattices $L_1, \ldots, L_k$ of total dimension n. The typical vector of L can be written

$$y = y_1 + y_2 + \cdots + y_k , \tag{1}$$

where each component y_i is in the subspace spanned by L_i, although not necessarily in L_i itself. What are the possibilities for $y_1, \ldots, y_k$?

The inner product of y_i with any vector of L_i is an integer, since it is the same as the inner product of y with that vector. This shows that y_i must be a member of the dual lattice L_i^*.

Plainly any y_i can be altered by adding a vector of L_i, so we may suppose that y_i is one of a standard system of representatives for the cosets of L_i in L_i^*. These representatives are called the *glue vectors* for L_i. It is usual to choose the glue vectors to be of minimal norm in their cosets. The quotient group L_i^* / L_i is called the *dual quotient*, or *glue group* for L_i. Its order is equal to the determinant of L_i.

So each possible lattice L is generated by $L_1 \oplus \cdots \oplus L_k$ together with certain vectors of the form (1) satisfying (a) each y_i is a glue vector for L_i, (b) the various vectors (1) have integral inner products with each other and (c) are closed under addition modulo $L_1 \oplus \cdots \oplus L_k$. We describe such a lattice L informally by saying that the *components* $L_1, \ldots, L_k$ have been *glued* together by the glue vectors (1). It may happen that there is a glue vector (1) in which only one y_i is nonzero, when we say that the component L_i has *self-glue*, and that y is a *self-glue* vector.

There is a standard choice of glue vectors [0], $[1], \ldots, [d-1]$ for each root lattice of determinant d, and we use $[a_1 \cdots a_k]$ as an abbreviation for the glue vector (1) in which $y_1 = [a_1]$ for L_1, $y_2 = [a_2]$ for L_2, ... The glue vectors and related properties of the root lattices are given in the Appendix.

In our applications of gluing theory it will usually be obvious that all automorphisms of L will permute the lattices $L_1, \ldots, L_k$ (often because $L_1 \oplus \cdots \oplus L_k$ will be the part of L generated by vectors of norms 1 and 2). In these circumstances there is a simple description of the automorphism group $G(L)$ of L.

The group of all permutations of the L_i that arise from automorphisms in $G(L)$ we shall call $G_2(L)$. It is isomorphic to the quotient group $G(L)/G_{01}$, where G_{01} consists of just those automorphisms that give the trivial permutation.

Let $G_0(L)$ be the normal subgroup of G_{01} consisting of those automorphisms which, for every i, send each glue vector y_i into a vector in the same coset $y_i + L_i$, i.e. which fix the glue vectors modulo the components. Then $G_{01}/G_0(L)$ is isomorphic to a permutation group acting on the glue vectors of each component: we call this permutation group $G_1(L)$. Thus the full group $G(L)$ is compounded of the groups $G_0(L)$, $G_1(L)$, $G_2(L)$, and has order

$$g(L) = g_0(L) g_1(L) g_2(L) , \tag{2}$$

where $g_i(L)$ is the order of $G_i(L)$. Also $G_0(L)$ is the direct product of the groups $G_0(L_i)$. But in general $G_1(L)$ is only a subgroup of the direct product of the $G_1(L_i)$ and therefore must be computed directly for each L.

If L is any integral lattice then we call the sublattice K generated by vectors of norms 1 and 2 the *Witt part* of L. If the Witt part is a direct sum

$$X_a \oplus Y_b \oplus Z_c \oplus \cdots$$

then we shall say that L has (*Kneser*) *shape*

$$X_a Y_b Z_c \cdots M_k ,$$

where $k = n - a - b - c - \cdots$ and M_k (the *remainder* lattice) is the sublattice of L orthogonal to the Witt part. If $X_a \oplus \cdots \oplus M_k$ has index r in L we shall often describe L by the symbol

$$(X_a Y_b Z_c \cdots M_k)^{+r} , \qquad (3)$$

and will omit r when its precise value does not concern us. This notation generalizes that of Coxeter (1951), except that we have added the plus sign for definiteness. Thus Coxeter's $A_n^r, D_n^r, \ldots$ are our $A_n^{+r}, D_n^{+r} \ldots$.

By definition M_k has minimal norm at least 3, and in the present paper its dimension is at most 3. We have adopted the following abbreviations for describing such lattices. A one-, two- or three-dimensional lattice is named by its Gram matrix, using the conventions

$$\begin{array}{llll} a_1 & \text{for} & (a), & \text{with generator } u, \\ (a^b c)_2 & \text{for} & \begin{bmatrix} a & b \\ b & c \end{bmatrix}, & \text{with generators } u, v, \\ (a^b c^f g^h)_3 & \text{for} & \begin{bmatrix} a & b & h \\ b & c & f \\ h & f & g \end{bmatrix}, & \text{with generators } u, v, w. \end{array} \qquad (4)$$

Glue vectors for these lattices are written as fractional linear combinations of their generator vectors. Thus

$$[\alpha\ \beta\ \gamma \cdot\cdot\cdot\ ] \text{ denotes } \alpha u + \beta v + \gamma w + \cdots . \tag{5}$$

Whenever possible we decompose the remainder lattice M_k as a direct sum. When the exact shape of M_k does not concern us, we use the convention that O_k denotes any k-dimensional lattice with minimal norm at least 3. Also $O^2_{\rightarrowtail}$ denotes the direct sum of two isomorphic lattices $O_{\rightarrowtail}$, and $O_{\rightarrowtail} O_m$ the direct sum of two nonisomorphic lattices.

One further convention is sometimes used when describing glue vectors. Parentheses in a glue vector indicate that all cyclic shifts of the parenthesized portion are included.

The following examples will illustrate the notation.

(i) The lattice $2_{14} = A_{13} 7_1\ [4\ {}^1\!/_7]$ is a 14-dimensional lattice of determinant 2 that contains a sublattice $A_{13} \oplus <u>$, where $(u, u) = 7$, and the glue is generated by $x \oplus y$, where x is the glue vector $[4] = (({}^2\!/_7)^{10}, (-\,{}^5\!/_7)^4)$ for A_{13}, and $y = {}^1\!/_7\, u$. The Witt part of this lattice is A_{13}, and its (Kneser) shape is $A_{13} 7_1$, or simply $A_{13} O_1$.

(ii) The four-dimensional lattice of determinant 21

$$21'_4 = A_1^2 (10^4 10)_2\ [1\ 0\ {}^1\!/_2\ 0,\ 0\ 1\ 0\ {}^1\!/_2]$$

contains a sublattice $A_1 \oplus A_1 \oplus M_2$, where $M_2 = <u, v>$ and u, v have Gram matrix

$$\begin{bmatrix} 10 & 4 \\ 4 & 10 \end{bmatrix} ,$$

and the glue is generated by $x \oplus 0 \oplus {}^1\!/_2\, u$ and $0 \oplus y \oplus {}^1\!/_2\, v$, where x is the glue vector [1] for the first A_1 and $y = [1]$ for the second A_1.

(iii) The lattice $1'_{16} = D_8^2[(12)]$ is the 16-dimensional unimodular lattice with Witt part $D_8^2 = D_8 \oplus D_8$. Generating glue vectors are [1] ⊕ [2] and [2] ⊕ [1]. The other nonzero glue vector is their sum, [3] ⊕ [3].

Finally we mention that, although D_n was so far defined only for $n \geq 4$, it is often convenient to extend the definition to $n \geq 1$, where we find

$$D_1 \cong 4_1 , \quad D_2 \cong A_1 \oplus A_1 , \quad D_3 \cong A_3 . \tag{6}$$

3. The genus

It is slightly easier, and more traditional, to discuss the genus using the language of quadratic forms rather than lattices. We follow the treatment in (Conway & Sloane, 1988, Chap. 15). Two integral quadratic forms are said to be in the same *genus* if they have the same signature and are equivalent over the p-adic integers for all primes p. We shall call integrally inequivalent forms (or lattices) f, g in the same genus *congeners*, and write $f \sim g$.

Any integral quadratic form f can be decomposed over the p-adic integers as a direct sum of terms qf_q, where each q is a power of p and each f_q has determinant prime to p. Our genus symbol for an n-dimensional form f has the shape

$$\mathrm{I}_n(\cdots) \text{ or } \mathrm{II}_n(\cdots) , \tag{7}$$

where the Roman numeral is the *Type* of f (I if the 2-adic summand f_1 has an odd diagonal entry, otherwise II) and the parentheses in (7) contain factors indicating the nontrivial Jordan summands for all p. For odd p a factor $q^{n\pm}$ indicates a summand qf_q with the Legendre symbol

$$\left[\frac{\det \ f_q}{p} \right] = \pm 1 \ .$$

For $p = 2$ a factor $q^{n \pm}$ indicates a summand qf_q for which f_q has Type II (i.e. the diagonal entries of f_q are even) and

$$\left[\frac{2}{\det \ f_q} \right] = \pm 1 \ ,$$

while $q^{\pm}_{+ \text{ or } -}$ indicates a one-dimensional summand (qd) with

$$\left[\frac{2}{d} \right] = \pm 1 \ , \quad \left[\frac{-1}{d} \right] = +1 \text{ or } -1$$

respectively.

For example, $(8 \ ^4 \ 8)_2 = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$ is written 4^{2-}, while the one-dimensional form 20_1 is 4^{-}_{+}, since

$$\left[\frac{2}{5} \right] = -1 \ , \quad \left[\frac{-1}{5} \right] = +1 \ .$$

For $p = 2$ there are many essentially different Jordan decompositions. For instance

$$(2) \oplus \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} \text{ , with symbol } 2^{+}_{+} \ 4^{2-} \ ,$$

is 2-adically equivalent to

$$(10) \oplus \begin{bmatrix} 8/5 & 4/5 \\ 4/5 & 32/5 \end{bmatrix} \text{ , with symbol } 2^{-}_{+} \ 4^{2+} \ .$$

For a complete description of these equivalences see (Conway & Sloane, 1988, Chap. 15, §7.8). In simple cases their effect is usually just to make some of the signs redundant, in which case we omit such signs from our genus symbols.

Our name for the typical lattice L_n of Table 1 is usually a fairly straightforward abbreviation of the genus symbol, omitting unnecessary information, except that we feel free to use the symbols

	d_n,	d'_n	d''_n
for	a unique,	unique Type I,	unique Type II

indecomposable n-dimensional lattice of determinant d (when such a lattice exists). Also, when the dual quotient L_n^* / L_n is cyclic, we attach the signs from the different p-adic symbols directly to the number d, in decreasing order of p, and omitting unnecessary final signs.

4. Enumeration of lattices of determinant 23 by the method of complementation

In this section we describe the first of our two basic methods of enumeration, the technique of ‘‘complementation’’, and use it to enumerate lattices of determinant $d = 23$. The idea is to glue the unknown lattice L to a low-dimensional lattice K of a genus which is in a sense complementary to the given one.

We say that two lattices K_m and L_n are *t-complementary* if their direct sum $K_m \oplus L_n$ has index t in an integral lattice M_{m+n}. If this is the case then

$$\det M_{m+n} = \frac{\det K_m \cdot \det L_n}{t^2} . \tag{8}$$

In particular, any lattice K_m of genus either $\mathrm{I}_m(23^+)$ or $\mathrm{II}_m(23^+)$ is 23-complementary to any lattice L_n of genus either $\mathrm{I}_n(23^-)$ or $\mathrm{II}_n(23^-)$, with $\det M_{m+n} = 1$. An unknown lattice L_n of genus $\mathrm{I}_n(23^-)$ or $\mathrm{II}_n(23^-)$ can therefore be found as the orthogonal complement $\mathrm{K}_m^\perp$ of the ''complementary'' lattice K_m of genus $\mathrm{I}_m(23^+)$ or $\mathrm{II}_m(23^+)$ in a unimodular ''mother'' lattice M_{m+n} (and of course the same statement is also true with the +'s and −'s interchanged).

We apply this with $K_m = 23_1$, of genus $\mathrm{I}_1(23^+)$, generated by a vector v say of norm 23, and with $K_m = 23_3$, of genus $\mathrm{I}_3(23^-)$, generated by three vectors a, b, c with Gram matrix

$$\begin{bmatrix} 5 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} , \tag{9}$$

and thus obtain the following result.

Proposition 1. *Any lattice* L_n *of genus* $\mathrm{I}_n(23^-)$ *or* $\mathrm{II}_n(23^-)$ *can be found as the lattice* $v^\perp$ *orthogonal to a vector* v *of norm 23 in a unimodular lattice* M_{n+1}. *Any lattice* L_n *of genus* $\mathrm{I}_n(23^+)$ *or* $\mathrm{II}_n(23^+)$ *can be found as the lattice* $K^\perp$ *orthogonal to a sublattice* $K = 23_3^- = \langle a, b, c \rangle$ *with Gram matrix (9) in a unimodular lattice* M_{n+3}.

We begin by finding the indecomposable lattices of genus $\mathrm{I}_n(23^-)$ or $\mathrm{II}_n(23^-)$. By referring to Table 1 we see that, for $n \le 7$, M_{n+1} must be I_{n+1} (since E_8 has no vector of norm 23). Then v is without loss of generality one of the following. (For by symmetry we may suppose the coordinates in v are nonnegative, and if any coordinate is zero then $v^\perp$ has a summand I_1 and so is decomposable.)

$$
\begin{array}{ll}
v & \text{Shape of } v^{\perp} \\
3\ 3\ 2\ 1 & A_1 O_2 \\
4\ 2\ 1\ 1\ 1 & A_2 O_2 \\
3\ 2\ 2\ 2\ 1\ 1 & A_2 A_1 O_2 \\
3\ 3\ 1\ 1\ 1\ 1\ 1 & A_4 A_1 O_1 \\
4\ 1\ 1\ 1\ 1\ 1\ 1\ 1 & A_6 O_1 \\
2\ 2\ 2\ 2\ 2\ 1\ 1\ 1 & A_4 A_2 O_1 \\
3\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 1 & A_5 A_1 O_2 \\
\cdots & \cdots
\end{array}
\tag{10}
$$

We illustrate how to find the precise structure of $L_n = v^{\perp}$ in the case $v = 4\ 2\ 1\ 1\ 1$. We first find the Witt part of $v^{\perp}$, and its Kneser shape (defined in §2). In this example $v^{\perp}$ contains two independent vectors *a* and *b* of norm 2:

$$
\begin{array}{rrrrrr}
v = & 4 & 2 & 1 & 1 & 1 \\
a = & 0 & 0 & 1 & -1 & 0 \\
b = & 0 & 0 & 0 & 1 & -1 \\
c = & 1 & -2 & 0 & 0 & 0 \\
d = & -1 & -1 & 2 & 2 & 2
\end{array} ,
\tag{11}
$$

and the Witt part of $v^{\perp}$ is the lattice A_2 of determinant 3 that they generate. Then the Kneser shape of $v^{\perp}$ is $A_2 M_2$, where the residual lattice M_2 is a two-dimensional sublattice of $v^{\perp}$ that is also orthogonal to this A_2, and contains no vectors of norm 1 or 2. It is easy to see that M_2 is generated by the vectors *c* and *d* in (11) (of norms 5 and 14, and inner product 1), so that

$$M_2 \cong (5\ {}^{1}\ 14)_2 ,$$

of determinant 69. It remains to find the glue. Since

$$\det\ (A_2 \oplus M_2) = 3 \cdot 69 = 3^2 \cdot 23 ,$$

$A_2 \oplus M_2$ has index 3 in $v^{\perp}$. In such a simple case it is easiest to guess the glue vectors. We observe that the glue vector $\frac{1}{3}(c + d)$ for M_2, which we abbreviate to $[\frac{1}{3}\ \frac{1}{3}]$, has integral inner product with each of *c* and *d*, and so belongs to M_2^*. This glue vector has norm

$$\frac{5 + 2 \cdot 1 + 14}{3^2} = \frac{7}{3} ,$$

and so can be glued to the glue vector [1], of norm $2/3$, for A_2. (The glue vectors for A_n, D_n, E_n are given in the Appendix.) So a possible gluing is

$$A_2 (5^{\,1}\ 14)_2 [1\ \tfrac{1}{3}\ \tfrac{1}{3}] \ .$$

But since our argument shows that $v^\perp$ is the *only* lattice of genus $\mathrm{I}_4 (23^-)$ or $\mathrm{II}_4 (23^-)$, this must be the structure of $v^\perp$.

Had there been any difficulty in guessing the glue vectors, we could have explicitly computed more generators. In this example it is easy to see that $v^\perp$ is generated by $A_2 \oplus M_2$ together with the vector $e = (0, -1, 1, 1, 0)$, which we resolve into the two 2-spaces of A_2 and M_2 as

$$e = (0, 0, \tfrac{1}{3}, \tfrac{1}{3}, - \tfrac{2}{3}) + (0, -1, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}) \ .$$

The first vector is the glue vector [1] for the A_2, and the second is $\tfrac{1}{3}(c + d)$.

The smallest lattice 23_m^- found in this way comes from $v = 3\ 3\ 2\ 1$, and is the lattice $23_3^- = \langle a, b, c \rangle$ defined by (9).

Next we use Proposition 1 to find the indecomposable lattices of genus $\mathrm{I}_n (23^+)$ or $\mathrm{II}_n (23^+)$, by looking for sublattices $K = 23_3^- = \langle a, b, c \rangle$ in a unimodular lattice M_{n+3}.

If $M_{n+3} = \mathrm{I}_{n+3}$ there are six possibilities for the vectors a, b, c:

$$\begin{array}{lcl} a & = & 2 \ + \ 0 \ 0 \\ b & = & + \ - \ + \ 0 \\ c & = & 0 \ 0 \ + \ + \ , \end{array}$$

$$\begin{array}{lcl} a & = & 2 \ + \ 0 \ 0 \ 0 \\ b & = & 0 \ + \ + \ + \ 0 \\ c & = & 0 \ 0 \ 0 \ + \ + \ , \end{array}$$

$$\begin{array}{lcl} a & = & + \ + \ + \ + \ + \\ b & = & + \ + \ - \ 0 \ 0 \\ c & = & + \ 0 \ 0 \ - \ 0 \ , \end{array}$$

$$\begin{array}{lcl} a & = & + \ + \ + \ + \ + \\ b & = & + \ + \ - \ 0 \ 0 \\ c & = & 0 \ 0 \ - \ + \ 0 \ , \end{array}$$

$$\begin{array}{lcl} a & = & + \ + \ + \ + \ + \ 0 \ 0 \\ b & = & 0 \ 0 \ 0 \ 0 \ + \ + \ + \\ c & = & 0 \ 0 \ 0 \ - \ + \ 0 \ 0 \ , \end{array}$$

$$\begin{array}{lcl} a & = & + \ + \ + \ + \ + \ 0 \ 0 \ 0 \\ b & = & 0 \ 0 \ 0 \ 0 \ + \ + \ + \ 0 \\ c & = & 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ + \ + \ . \end{array}$$

In the first four cases $K^{\perp}$ is a one- or two-dimensional lattice, namely

$$23_1 , \ (3^1 8)_2 , \ (4^1 6)_2 , \ \text{and} \ (2^1 12)_2$$

— we omit the analysis, since these lattices are more easily obtained by Gauss' theory of reduced forms (see §7).

In the fifth case $K^{\perp}$ contains the vectors

$$\begin{array}{lcl} d & = & + \ - \ 0 \ 0 \ 0 \ 0 \ 0 \\ e & = & 0 \ + \ - \ 0 \ 0 \ 0 \ 0 \\ f & = & 0 \ 0 \ 0 \ 0 \ 0 \ + \ - \ , \end{array}$$

generating $A_2 \oplus A_1$, and the shortest vector in $K^{\perp}$ perpendicular to d, e, f is

$$g = 4\ 4\ 4\ {-6}\ {-6}\ 3\ 3\ ,$$

of norm 138. So $K^\perp$ is

$$A_2 A_1\, 138_1 [1\ 1\ {}^1\!/\!_6]$$

(the glue vector $[1\ 1\ {}^1\!/\!_6]$ being found as before). The last possibility can be analyzed similarly, and leads to

$$A_3 (3^1 31)_2 [1\ {}^1\!/\!_4\ {}^1\!/\!_4]\ .$$

If $n \le 7$, so that $n + 3 \le 10$, the only other possibilities for M_{n+3} are $\mathrm{I}_1 \oplus E_8$ and $\mathrm{I}_2 \oplus E_8$ (since E_8 itself has Type II). These yield four more lattices, namely

$$D_8 (8^2 12)_2 [1\ {}^1\!/\!_2\ 0,\ 2\ 0\ {}^1\!/\!_2] \text{ and } D_5\ 92_1 [1\ {}^1\!/\!_4] \qquad (12)$$

if $M_9 = \mathrm{I}_1 \oplus E_8$, and

$$E_6\ 69_1 [1\ {}^1\!/\!_3] \text{ and } A_4 A_1 (14^6 19)_2 [1\ 1\ {}^1\!/\!_{10}\ {}^3\!/\!_5] \qquad (13)$$

if $M_{10} = \mathrm{I}_2 \oplus E_8$. The analysis of these cases involves heavy knowledge of the action of the automorphism group of E_8 on various configurations of vectors. We illustrate by showing how the first lattice in (13) was found.

We must first find $K = 23_3^- = \langle a, b, c \rangle$ in $\mathrm{I}_2 \oplus E_8$. One of the two possibilities is to take

$$\begin{aligned} a &= 2\ 1;\ 0\ \ \ldots\ 0 \\ b &= 0\ 1;\ b_1\ \ldots\ b_8 \\ c &= 0\ 0;\ c_1\ \ldots\ c_8\ . \end{aligned}$$

Then from (9) the vectors $b_1 \ldots b_8$ and $c_1 \ldots c_8$ must generate a sublattice A_2 of E_8. Since the automorphism group of E_8 is known to be transitive on sublattices A_r for $0 \le r \le 8$, we may choose

$$\begin{aligned} a &= 2\ +;\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ b &= 0\ +;\ +\ 0\ 0\ 0\ 0\ 0\ 0\ + \\ c &= 0\ 0;\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ , \end{aligned}$$

and then proceeding as before we find that $K^{\perp} = E_6\ 69_1 [1\ \tfrac{1}{3}]$.

The following table summarizes the indecomposable n-dimensional lattices of determinant 23 for $n \le 7$.

n	Lattices
1	23_1
2	$A_1 46_1 [1\ \tfrac{1}{2}]$, $(4^1 6)_2$, $(3^1 8)_2$
3	$A_1 (5^2 10)_2\ [1\ 0\ \tfrac{1}{2}]$
4	$A_2 A_1 138_1\ [1\ 1\ \tfrac{1}{6}]$, $A_2 (5^1 14)_2\ [1\ \tfrac{1}{3}\ \tfrac{1}{3}]$
5	$A_3 (3^1 31)_2\ [1\ \tfrac{1}{4}\ \tfrac{1}{4}]$, $A_2 A_1 (7^3 21)_2\ [1\ 1\ \tfrac{1}{2}\ \tfrac{1}{6}]$
6	$A_4 A_1 230_1\ [2\ 1\ \tfrac{1}{10}]$, $D_5 92_1\ [1\ \tfrac{1}{4}]$, $D_4 (8^2 12)_2\ [1\ \tfrac{1}{2}\ 0,\ 2\ 0\ \tfrac{1}{2}]$
7	$E_6 69_1\ [1\ \tfrac{1}{3}]$, $A_6 161_1\ [3\ \tfrac{1}{7}]$, $A_4 A_2 345_1\ [1\ 1\ \tfrac{1}{15}]$, $A_4 A_1 (14^6 19)_2\ [1\ 1\ \tfrac{1}{10}\ \tfrac{3}{5}]$

Determinant 19. A similar analysis show that the indecomposable lattices of determinant 19 in seven dimensions are

$$19_7^- = A_5 O_2 = A_5\ (10^{\,4}\ 13)_2\ [1\ \tfrac{1}{6}\ \tfrac{1}{3}]\ ,$$

obtained from $v^{\perp}$ in I_8, and

$$19_7^+ = D_5 O_2 = D_5\ (5^{\,2}\ 16)_2\ [1\ \tfrac{1}{2}\ \tfrac{1}{4}]\ ,$$

obtained from $< a,\ b >^{\perp}$ in $\mathrm{I}_1 \oplus E_8$, where v, a, b are the vectors

$$\begin{aligned} v &= 3\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ , \\ a &= 2\ ;\ 2\ +\ +\ 0\ 0\ 0\ 0\ 0\ , \\ b &= 0\ ;\ +\ -\ 0\ 0\ 0\ 0\ 0\ 0\ . \end{aligned}$$

$< v >$ has genus $\mathrm{I}_1 (19^+)$ and $< a,\ b >$ has genus $\mathrm{I}_2 (19^-)$.

5. Enumeration of lattices of determinant 24 by the method of supplementation

In this section we describe the method of ‘‘supplementation’’ and use it to enumerate lattices of determinant $d = 24$.

Proposition 2. *Any lattice L_n of determinant $4d$ is a sublattice of index 2 in a lattice M_n of determinant d.*

Proof. The dual quotient group L_n^* / L_n has a subgroup of order 4. If this is cyclic let v be an element of order 4. Then $2v \cdot 2v = 4v \cdot v \in L_n \cdot L_n^* \subseteq \mathbf{Z}$, so $M_n = \langle L_n , 2v \rangle$ is integral and contains L_n as a sublattice of index 2.

Otherwise L_n^* / L_n contains a four-group $\{0, u, v, w\}$ with $u + v + w = 0$. The norms $N(u)$, $N(v)$, $N(w)$ are in $\frac{1}{2} \mathbf{Z}$, since $2N(u) = 2u \cdot u \in \mathbf{Z}$. They cannot all be congruent to $\frac{1}{2}$ modulo 1, since

$$
\begin{aligned}
0 = N(u + v + w) &= N(u) + N(v) + N(w) + 2(u \cdot v + v \cdot w + w \cdot u) \\
&\equiv N(u) + N(v) + N(w) \qquad (\mathrm{mod}\ 1)\ .
\end{aligned}
$$

If say $N(u) \in \mathbf{Z}$ then we may take $M_n = \langle L_n , u \rangle$.

Proposition 2 implies that we can write L_n as the kernel of a linear map from M_n to $\mathbf{Z}/2\mathbf{Z}$. Now any linear map from M_n to $\mathbf{Z}$ is a map

$$
x \rightarrow x \cdot v \qquad (x \in M_n)
$$

for some $v \in M_n^*$. Thus we have established:

Proposition 3. *Any lattice L_n of determinant 24 is a sublattice $v^{\perp 2}$ of a lattice M_n of determinant 6, where $v \in M_n^*$ and*

$$
v^{\perp 2} = \{ x \in M_n : x \cdot v \equiv 0 \quad (mod\ 2) \}\ .
$$

Since $v^{\perp 2}$ is unchanged if a member of $2M_n^*$ is added to v, the value of v is only important modulo $2M_n^*$. Moreover, if M_n is a nontrivial direct sum of several lattices, the coordinates of v must not vanish in any summand.

Examination of Table 1 shows that to find all L_n with $n \le 7$ we must consider these possibilities for M_n:

$$\begin{array}{ll} \mathrm{I}_m \oplus 6_1, & \mathrm{I}_m \oplus A_1 \oplus 3_1, \\ \mathrm{I}_m \oplus A_1 \oplus A_2, & \mathrm{I}_m \oplus A_5, \\ E_6 \oplus A_1, & D_6\ 6_1[1\ {}^1\!/_2]. \end{array}$$

Suppose $M_n = \mathrm{I}_m \oplus 6_1$ $(m = n - 1)$, the lattice of vectors

$$(x_1 \ldots x_m;\ y\sqrt{6}) ,$$

where $x_1, \ldots, x_m, y \in \mathbf{Z}$. The only $v \in M_n^*$ for which no coordinate vanishes when we work modulo $2M_n^*$ is

$$v = (1\ 1 \ldots 1;\ {}^1\!/_2\sqrt{6})$$

and then $v^{\perp 2}$ is (assuming $m \ge 4$)

$$D_m\ 24_1[2\ {}^1\!/_2]\ . \qquad (14)$$

(The generator of 24_1 is $(0 \ldots 0;\ 2\sqrt{6})$ and the glue vector is $(0 \ldots 0\ 1;\ \sqrt{6})$.) For smaller values of m the lattice (14) is

$$\begin{array}{ll} 24_1 \text{ (if } m = 0), & (4^2\ 7)_2 \text{ (if } m = 1), \\ A_1^2\ 24_1\ [1\ 1\ {}^1\!/_2] \text{ (if } m = 2), & A_3\ 24_1\ [2\ {}^1\!/_2] \text{ (if } m = 3) \end{array}$$

(compare Eq. (6)). Similar modifications for small m must be made to the lattices (15)-(19) below — the results will be found in the summary table at the end of the section.

In a similar way we obtain

$$D_m (5^1 5)_2 [2\ {}^1/_2\ {}^1/_2] \text{ from } M_n = \mathrm{I}_m \oplus A_1 \oplus 3_1 \ , \qquad (15)$$

$$D_m A_1 6_1 8_1 [2\ 0\ 0\ {}^1/_2,\ 0\ 1\ {}^1/_2\ {}^1/_2] \text{ from } M_n = \mathrm{I}_m \oplus A_1 \oplus A_2 \ . \qquad (16)$$

We next consider $M_n = \mathrm{I}_m \oplus A_5$ ($m = n - 5$). Now $A_5^* / 2 A_5^*$ has order 32 with

	representative						number
$v_0 =$	(0	0	0	0	0	0)	(1)
$v_1 =$	$(-{}^5/_2$	$^1/_2$	$^1/_2$	$^1/_2$	$^1/_2$	$^1/_2)$	(6)
$v_2 =$	(+	$-$	0	0	0	0)	(15)
$v_3 =$	$(^1/_2$	$^1/_2$	$^1/_2$	$-{}^1/_2$	$-{}^1/_2$	$-{}^1/_2)$	(10)

So v must be

$$v = (1\ 1 \ldots 1;\ v_i), \qquad i = 1, 2, 3 \ .$$

In the three cases $v^{\perp 2}$ is

$$D_m A_4 120_1 [2\ 1\ {}^1/_{10}] \ , \qquad (17)$$

$$D_m A_3 A_1 12_1 [2\ 1\ 1^1/_4] \ , \qquad (18)$$

$$D_m A_2^2 24_1 [2\ 1\ 1\ {}^1/_6] \ , \qquad (19)$$

respectively.

If $M_n = E_6 \oplus A_1$ there are two cases:

$$v = v_{4/3} + [1] \text{ and } v = v_2 + [1] \ ,$$

where $v_{4/3}$ and v_2 are elements of norms $^4/_3$ and 2 in E_6^*, and [1] is the glue vector for A_1, yielding

$$D_5\ 8_1\ 12_1 [1\ {}^1/_2\ {}^1/_4] \text{ and } A_5\ A_1\ 8_1 [3\ 1\ {}^1/_2]$$

respectively.

We omit the analysis of the final possibility for M_n, $D_6\ 6_1 [1\ \frac{1}{2}]$, which yields five further lattices

$$D_5\ 4_1\ 24_1 [1\ \tfrac{1}{4}\ \tfrac{1}{4}]\ ,\qquad A_5\ 6_1\ 24_1 [1\ \tfrac{1}{2}\ \tfrac{1}{3}]\ ,$$
$$A_5\ 6_1\ 24_1 [1\ \tfrac{1}{2}\ \tfrac{1}{6}]\ ,\qquad A_5\ 3_1\ 12_1 [2\ \tfrac{1}{3}\ \tfrac{1}{3}]\ ,$$
$$A_3^2\ 24_1 [1\ 1\ \tfrac{1}{4}]\ .$$

The following table summarizes the indecomposable *n*-dimensional lattices of determinant 24 for $n \le 7$.

n	Lattices
1	24_1
2	$(4^2 7)_2, (5^1 5)_2$
3	$A_1^2\ 24_1\ [1\ 1\ \frac{1}{2}], A_1 6_1 8_1\ [1\ \frac{1}{2}\ \frac{1}{2}], (3^1 4^1 3^{-1})_3$
4	$A_3\ 24_1\ [2\ \frac{1}{2}], A_1^2 (5^1 5)_2\ [1\ 1\ \frac{1}{2}\ \frac{1}{2}], A_1\ 6_1 (3^1 3)_2\ [1\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]$
5	$D_4 24_1 [2\ \frac{1}{2}], A_4\ 120_1 [2\ \frac{1}{5}], A_3 (5^1 5)_2\ [2\ \frac{1}{2}\ \frac{1}{2}], A_2^2 24_1\ [1\ 1\ \frac{1}{3}],$ $A_1^3 6_1 8_1 [1\ 1\ 0\ 0\ \frac{1}{2}\ ,\ 0\ 0\ 1\ \frac{1}{2}\ \frac{1}{2}]$
6	$D_5 24_1 [2\ \frac{1}{2}], D_4 (5^1 5)_2\ [2\ \frac{1}{2}\ \frac{1}{2}], A_4 (4^2 31)_2\ [1\ \frac{2}{5}\ \frac{1}{5}], A_3 A_1\ 4_1\ 12_1\ [1\ 1\ \frac{1}{2}\ \frac{1}{4}],$ $A_3 A_1 6_1 8_1 [2\ 0\ 0\ \frac{1}{2},\ 0\ 1\ \frac{1}{2}\ \frac{1}{2}]\ , A_2^2 (4^2 7)_2 [1\ 1\ \frac{1}{3}\ \frac{1}{3}]$
7	$D_6 24_1 [2\ \frac{1}{2}], D_5 4_1 24_1 [1\ \frac{1}{4}\ \frac{1}{4}], D_5 8_1 12_1 [1\ \frac{1}{2}\ \frac{1}{4}], D_5 (5^1 5)_2\ [2\ \frac{1}{2}\ \frac{1}{2}], A_5 A_1 8_1\ [3\ 1\ \frac{1}{2}],$ $A_5 3_1 12_1 [2\ \frac{1}{3}\ \frac{1}{3}], A_5 6_1 24_1 [1\ \frac{1}{2}\ \frac{1}{6}], A_5 6_1 24_1 [1\ \frac{1}{2}\ \frac{1}{3}], D_4 A_1 6_1 8_1 [2\ 0\ 0\ \frac{1}{2},\ 0\ 1\ \frac{1}{2}\ \frac{1}{2}],$ $A_4 A_1^2 120_1 [1\ 1\ 1\ \frac{1}{10}]\ , A_3^2 24_1 [1\ 1\ \frac{1}{4}], A_3 A_1^3 12_1 [1\ 1\ 1\ 1\ \frac{1}{4}], A_2^2 A_1^2 24_1 [1\ 1\ 1\ 1\ \frac{1}{6}]$

6. Enumeration of lattices of determinant 25 using both methods

In this section we find the indecomposable lattices of determinant 25 and dimension $n \le 7$.

Proposition 4. *Any lattice L_n of determinant d divisible by p^2, p prime ≥ 3, is either of index p in a lattice M_n of determinant d/p^2, or else the p-part of the genus symbol is p^{2-} if $p \equiv 1$ (mod 4) or p^{2+} if $p \equiv -1$ (mod 4).*

Proof. If the dual quotient group L_n^* / L_n contains an element v of order p^2 then (as in the proof of Proposition 2 for the case $p = 2$) we may take $M_n = < L_n , pv >$.

Otherwise the p-part of L_n^* / L_n can be regarded as a vector space on which the map

$$x \rightarrow x \cdot x \qquad (\mathrm{mod}\ 1)$$

is a quadratic form. If this space contains a nonzero isotropic vector v then we may take $M_n = < L_n , v >$. This certainly happens if p^3 divides the determinant, and when $p^2 \parallel d$ it fails just in the indicated cases (when the quadratic form has Witt defect 1). When it does fail we fall back on the method of complementation.

In particular, when $p = 5$, if the unknown lattice L_n has genus $\mathrm{I}_n (5^{2-})$ or $\mathrm{II}_n (5^{2-})$ it is 5^2-complementary to any particular such lattice, for example to the lattice $K_3 = 5_1 \oplus 5_2$ generated by vectors a, b, c with Gram matrix

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} . \qquad (20)$$

Thus we have the following result.

Proposition 5. *Any lattice L_n of determinant 25 is either of index 5 in a unimodular lattice M_n, or is the lattice $K^{\perp}$ orthogonal to a sublattice $K = 5_1 \oplus 5_2 = < a, b, c >$ with Gram matrix (20) in a unimodular lattice M_{n+3}. In the first case L_n can be written as $L_n = v^{\perp 5}$ in M_n, where $v \in M_n^*$ and*

$$v^{\perp 5} = \{ x \in M_n : x \cdot v \equiv 0 \quad (mod\ 5) \} .$$

The second case occurs just when the genus of L_n is $\mathrm{I}_n (5^{2-})$ or $\mathrm{II}_n (5^{2-})$.

In the first case (supplementation), when $n \le 7$ we need only consider $M_n = \mathrm{I}_n$. The vector v can be taken without loss of generality to be

$$v = 1^i 2^j \ , \ \ i + j = n \ ,$$

since there are symmetries changing the signs of the coordinates. The resulting lattice L_n will be denoted by Ψ_{ij}. Since the coordinates of $2v$ are congruent modulo 5 to $2^i (-1)^j$, we see that $\Psi_{ij} \cong \Psi_{ji}$. We also set $\Psi_i = \Psi_{i0}$. There are now nineteen cases with $n \le 7$, but

$$\Psi_{11} = 5_1 \oplus 5_1 , \ \Psi_{22} = 5_2 \oplus 5_2 , \ \Psi_5 = 5_1 \oplus 5_4$$

are decomposable. The other sixteen lattices are listed below.

Second, to use complementation, we must take M_{n+3} to be I_{n+3}, $\mathrm{I}_1 \oplus E_8$ or $\mathrm{I}_2 \oplus E_8$ (since $n \le 7$), and find all sublattices $K = 5_1 \oplus 5_2 = <a, b, c>$ with Gram matrix (20). Noting that $<-a, b, c> = <a, b, c>$ and that L_n is decomposable if $<a>$ and $<b, c>$ lie in disjoint direct summands of M_{n+3}, we obtain only the following cases:

$$\begin{array}{lcllllll} a & = & + & + & + & + & + & 0 \\ b & = & + & - & 0 & 0 & 0 & + \\ c & = & + & 0 & - & 0 & 0 & 0 \end{array} \qquad \text{in } \mathrm{I}_6, \text{ yielding } 5_1 \oplus 5_2,$$

$$\begin{array}{lclllllll} a & = & + & + & + & + & + & 0 & 0 \\ b & = & + & - & 0 & 0 & 0 & + & 0 \\ c & = & 0 & 0 & 0 & 0 & 0 & + & + \end{array} \qquad \text{in } \mathrm{I}_7, \text{ yielding } A_2 (10^5 10)_2 \, [1 \ \tfrac{1}{3} \ \tfrac{1}{3}],$$

$$\begin{array}{lclllllllll} a & = & +; & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & = & -; & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} \\ c & = & 0; & 0 & + & + & 0 & 0 & 0 & 0 & 0 \end{array} \qquad \text{in } \mathrm{I}_1 \oplus E_8, \text{ yielding } 5_2 \oplus 5_4,$$

$$\begin{array}{l} a = 0\ +;\ 2\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ b = +\ 0;\ 0\ +\ -\ 0\ 0\ 0\ 0\ 0 \\ c = 0\ 0;\ 0\ +\ 0\ -\ 0\ 0\ 0\ 0 \end{array} \qquad \text{in } I_2 \oplus E_8 \text{, yielding}$$

$$D_4(7^1\ 7^1\ 7^{-3})_3[1\ {}^1/_2\ {}^1/_2\ 0,\ 3\ 0\ {}^1/_2\ {}^1/_2]\ .$$

The following table summarizes the indecomposable n-dimensional lattices of determinant 25 for $n \le 7$.

n	Lattices
1	$25_1 = \Psi_1$
2	$A_1 50_1 [1\ {}^1/_2] = \Psi_2$
3	$A_2 75_1 [1\ {}^1/_3] = \Psi_3, A_1(3^1 17)_2\ [1\ {}^1/_2\ {}^1/_2] = \Psi_{21}$
4	$A_3 100_1 [1\ {}^1/_4] = \Psi_4, A_2(4^1 19)_2\ [1\ {}^1/_3\ {}^2/_3] = \Psi_{31}, A_2(10^5 10)_2\ [1\ {}^1/_3\ {}^1/_3]$
5	$A_3(8^2 13)_2\ [1\ {}^1/_4\ {}^1/_2] = \Psi_{41}, A_2 A_1(11^2 14)_2\ [1\ 1\ {}^1/_3\ {}^1/_6] = \Psi_{32}$
6	$A_5 150_1 [1\ {}^1/_6] = \Psi_6,\ A_4 5_1 25_1 [2\ {}^2/_5\ {}^1/_5] = \Psi_{51}, A_3 A_1(6^2 34)_2[1\ 0\ {}^3/_4\ {}^1/_4,\ 2\ 1\ {}^1/_2\ 0] = \Psi_{42},\ A_2^2 15_1^2 [1\ 1\ {}^1/_3\ 0,\ 1\ 2\ 0\ {}^1/_3] = \Psi_{33}$
7	$A_6 175_1 [2\ {}^1/_7] = \Psi_7,\ A_5 10_1 15_1 [1\ {}^1/_2\ {}^1/_3] = \Psi_{61},\ D_4(7^1 3^1 7^{-3})_3\ [1\ {}^1/_2\ {}^1/_2\ 0,\ 3\ 0\ {}^1/_2\ {}^1/_2]$, $A_4 A_1 5_1 50_1 [1\ 1\ {}^1/_5\ {}^3/_{10}] = \Psi_{52}, A_3 A_2(7^1 43)_2 [1\ 1\ {}^1/_{12}\ {}^5/_{12}] = \Psi_{43}$

7. The tables

Table 0 contains a summary. The typical entry gives the Kneser shapes of the indecomposable lattices of determinant d and dimension n, or just the number of such lattices (in parentheses). A star indicates a dimension in which there are two or more indecomposable congeners. For the continuation of the $d = 1$ column see Borcherds (1984, 1988), Conway & Sloane (1982, 1982b, and 1988, Table 16.7), and for the continuation of the $d = 2$ and 3 columns see (Conway & Sloane, 1988, Tables 15.8, 15.9).

Table 1 gives all indecomposable n-dimensional lattices L_n of determinant $d \leq 23$ below the critical dimension. The first column gives our name for L_n (described below), and subsequent columns give its Kneser shape (see §2), generators for the glue (using the conventions described in §2), its genus (see §3), and the values of $g_1 = g_1(L_n)$ and $g_2 = g_2(L_n)$ (see Eq. (2)). We note that $g_0(L_n) = 1$ for all the lattices O_1, O_2, O_3 that appear in this table.

The lattices were found by the methods described in §§4, 5, 6, except that for dimension 2 we used Gauss' theory of reduced forms (see (Gauss, 1801), (Edwards, 1977), (Conway & Sloane, 1988, Chap. 15)).

Table 2 gives the lattices of determinant d in the critical dimension.

Appendix. Properties of root lattices

In this Appendix we list the glue vectors, glue groups and automorphism groups for the root lattices. We make very extensive use of this information. Further properties of these lattices will be found in (Conway & Sloane, 1987, especially Chap. 4).

The typical glue vector for A_n $(n \geq 1)$ is

$$[i] = \left[\frac{i}{n+1}, \; \ldots, \; \frac{i}{n+1}, \; \frac{-j}{n+1}, \; \ldots, \; \frac{-j}{n+1} \right],$$

with j components equal to $i/(n+1)$, and i components equal to $-j/(n+1)$, where $i + j = n + 1$ and $0 \leq i \leq n$. The norm of $[i]$ is $ij/(n+1)$. The glue group is cyclic of order $n+1$, with addition $[j] + [k] = [j+k]$. Automorphism group: G_0 is the Weyl group of A_n, which is the symmetric group S_{n+1} of all permutations of the coordinates, and G_1 is the group of order 2 generated by the negation of all coordinates (which interchanges $[i]$ and $[n+1-i]$), except that $G_1 = 1$ when $n = 1$.

The glue vectors for D_n ($n \geq 4$) are

$$\begin{array}{ll} [0] = (0, 0, \ldots, 0), & \text{norm } 0, \\ [1] = (\tfrac12, \tfrac12, \ldots, \tfrac12), & \text{norm } n/4, \\ [2] = (0, 0, \ldots, 1), & \text{norm } 1, \\ [3] = (\tfrac12, \tfrac12, \ldots, -\tfrac12), & \text{norm } n/4 . \end{array}$$

The glue group is a 4-group ($[i] + [i] = 0$) if n is even, or a cyclic group of order 4 ($[1] + [2] = [3]$) if n is odd. Automorphism group:

$$\begin{array}{llll} n = 4: & g_0 = 2^3 \cdot 4!, & g_1 = 3! & \text{(all perms of } [1], [2], [3]), \\ n \neq 4: & g_0 = 2^{n-1} \cdot n!, & g_1 = 2 & \text{(interchange } [1] \text{ and } [3]). \end{array}$$

G_0 is generated by all permutations together with sign changes of evenly many coordinates, and G_1 contains the sign change of the last coordinate and, for $n = 4$ only, the Hadamard matrix

$$H_4 = \tfrac12 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} .$$

The only glue vector for E_8 is $[0] = (0^8)$, since $E_8^* = E_8$. Automorphism group: $g_0 = 2^{14}\ 3^5\ 5^2\ 7 = 696729600$, $g_1 = 1$. G_0 is the Weyl group $W(E_8)$, generated by all permutations of 8 letters, all even sign changes, and the matrix diag $\{H_4, H_4\}$.

The glue vectors for E_7 are

$$\begin{array}{ll} [0] = (0, 0, 0, 0, 0, 0, 0, 0), & \text{of norm } 0, \\ [1] = (\tfrac14, \tfrac14, \tfrac14, \tfrac14, \tfrac14, \tfrac14, -\tfrac34, -\tfrac34), & \text{of norm } \tfrac32 . \end{array}$$

The glue group is cyclic of order 2. Automorphism group: G_0 is the Weyl group $W(E_7)$, of order $g_0 = 2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2903040$, $g_1 = 1$.

The glue vectors for E_6 are

$$
\begin{aligned}
[0] &= (0; 0, 0, 0, 0, 0, 0; 0) , && \text{of norm } 0 , \\
[1] &= (0; -{}^2/_3, -{}^2/_3, {}^1/_3, {}^1/_3, {}^1/_3, {}^1/_3; 0) , && \text{of norm } {}^4/_3 , \\
[2] &= -[1] , && \text{of norm } {}^4/_3 .
\end{aligned}
$$

The glue group is cyclic of order 3. Automorphism group: G_0 is the Weyl group $W(E_6)$, of order $g_0 = 2^7 \cdot 3^4 \cdot 5 = 51840$, G_1 is cyclic of order 2 (generated by negation).

List of Table Captions

Table 0. Summary table, showing Kneser shapes of indecomposable lattices of determinant d and dimension n. The symbol (k) indicates that there are k lattices, and $(k)^*$ the presence of congeners.

Table 1. Indecomposable lattices of determinant $d \leq 23$ below the critical dimension. An n-dimensional lattice of determinant d is named d_n, where d may be factorized, and possibly have signs or primes attached. For full explanation of the notation see the text.

Table 2. Supplementary table, giving indecomposable lattices of determinant $d \leq 23$ in the critical dimension n, with congeners indicated by ~.

Table 0.

n	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$
1	I_1	A_1	O_1	O_1	O_1	O_1	O_1
2	—	—	A_2	—	$A_1 O_1$	—	$A_1 O_1$
3	—	—	—	A_3	—	—	$A_2 O_1$
4	—	—	—	D_4	A_4	—	—
5	—	—	—	D_5	—	A_5	—
6	—	—	E_6	D_6	—	—	A_6 $D_5 O_1$
7	—	E_7	—	D_7	$E_6 O_1$	$D_6 O_1$	—
8	E_8	—	$E_7 O_1$	D_8	$E_7 O_1$ $D_7 O_1$	—	$E_7 O_1$ $E_6 A_1 O_1$
9	—	—	—	D_9 $D_8 O_1$ $E_7 A_1 O_1$	—	$D_7 A_1 O_1$	$A_8 O_1$ $E_7 O_2$
10	—	—	$D_9 O_1$	D_{10} $D_8 A_1^2$ $E_7 A_1^3$	$A_9 A_1$ $E_7 A_2 O_1$	$A_9 O_1$	$D_9 O_1$ $D_8 O_2$ $D_7 A_2 O_1$

Table 0 (cont'd)

n	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$
11	—	$D_{10}A_1$	A_{11}	D_{11} D_8A_3 $E_7A_3A_1$	$A_{10}O_1$ D_9O_2	$(4)^*$	$A_9A_1O_1$ $E_6A_4O_1$
12	D_{12}	—	$D_{10}A_1O_1$ E_7A_5	$(6)^*$	$(4)^*$		$(8)^*$
13	—	E_7D_6	$A_{12}\ O_1$ $E_6^2\ O_1$				
14	E_7^2	$A_{13}O_1$	$(4)^*$				
15	A_{15}	$D_{14}A_1$ $D_8D_6A_1$					
16	D_{16} D_8^2	$(2)^*$					
17	$A_{11}E_6$						
18	$(4)^*$						

Table 0 (cont'd)

n	$d = 8$	$d = 9$	$d = 10$	$d = 11$	$d = 12$	$d = 13$	$d = 14$
1	O_1	O_1	O_1	O_1	O_1	O_1	O_1
2	O_2	$A_1 O_1$	—	$A_1 O_1$ O_2	O_2	$A_1 O_1$	O_2
3	$A_1^2 O_1$	—	$A_2 O_1$	—	$A_1^2 O_1$ $A_1 O_1 O_1$	$A_2 O_1$ $A_1 O_2$	—
4	$A_3 O_1$	$A_3 O_1$	—	$A_2 A_1 O_1$	$A_3 O_1$ $A_1^3 O_1$	$A_3 O_1$	—
5	$D_4 O_1$	$A_4 O_1$	—	$A_4 O_1$	$D_4 O_1$ $A_3 A_1 O_1$	—	$A_4 O_1$ $A_3 A_1 O_1$
6	$D_5 O_1$	$A_5 O_1$	$A_5 O_1$	$D_5 O_1$	$D_5 O_1$ $D_4 A_1 O_1$	$A_5 O_1$ $A_4 A_1 O_1$	—
7	A_7 $E_6 O_1$ $D_6 O_1$	$A_6 O_1$	$D_6 O_1$ $D_5 A_1 O_1$	$E_6 O_1$ $A_6 O_1$	(4)	$D_5 O_2$	$E_6 O_1$ $D_6 O_1$ $A_5 A_1 O_1$
8	(4)	$(5)^*$	—	$(3)^*$	(6)	$(5)^*$	$D_6 O_2$ $A_6 O_1 O_1$
9	(6)		$(4)^*$		$(11)^*$		$(4)^*$
10	$(6)^*$						

Table 0 (cont'd)

n	$d = 15$	$d = 16$	$d = 17$	$d = 18$	$d = 19$	$d = 20$
1	O_1	O_1	O_1	O_1	O_1	O_1
2	$A_1 O_1$ O_2	O_2	$A_1 O_1$ O_2	—	$A_1 O_1$ O_2	O_2 O_2
3	—	$A_2 O_1$ $A_1^2 O_1$ O_3	$A_1 O_2$	$A_1 O_2$	$A_2 O_1$ $A_1 O_2$	$A_1^2 O_1$ $A_1 O_1 O_1$ O_3
4	$A_2 O_2$	$A_3 O_1$ $A_2 O_2$ $A_1^2 O_1^2$	$A_3 O_1$ $A_2 A_1 O_1$	$A_2 O_1 O_1$	$A_1^2 O_2$	$A_3 O_1$ $A_1^3 O_1$ $A_1^2 O_2$
5	$A_2^2 O_1$	(5)	$A_3 O_2$	$A_3 O_1 O_1$	$A_4 O_1$	(5)
6	$D_5 O_1$ $A_5 O_1$ $D_4 O_2$	(7)	$A_4 A_1 O_1$	$A_4 O_2$	$(4)^*$	(5)
7	$A_6 O_1$ $A_5 O_1 O_1$	(9)	$(3)^*$	$(5)^*$	$D_5 O_2$ $A_5 O_2$	$(8)^*$
8	$(5)^*$	$(13)^*$				

Table 0 (cont'd)

n	$d = 21$	$d = 22$	$d = 23$	$d = 24$	$d = 25$
1	O_1	O_1	O_1	O_1	O_1
2	$A_1 O_1$ O_2	—	(3)*	O_2 O_2	$A_1 O_1$
3	O_3	$A_2 O_1$ $A_1 O_2$	$A_1 O_2$	$A_1^2 O_1$ $A_1 O_1 O_1$ O_3	$A_2 O_1$ $A_1 O_2$
4	$A_3 O_1$ $A_1^2 O_2$	$A_2 O_2$	$A_2 A_1 O_1$ $A_2 O_2$	$A_3 O_1$ $A_1^2 O_2$ $A_1 O_1 O_2$	$A_3 O_1$ $A_2 O_2$ $A_2 O_2$
5	$A_4 O_1$	$A_3 A_1 O_1$	$A_3 O_2$ $A_2 A_1 O_2$	(5)	$A_3 O_2$ $A_2 A_1 O_2$
6	(4)	$A_5 O_1$ $A_4 O_2$	(3)*	(6)	(4)*
7	$D_5 O_2$ $A_5 O_2$ $A_5 O_2$	(4)*	(4)*	(13)*	(5)*
8	(9)*				

Table 1.

Name	Shape	Glue	Genus	$g_1 \cdot g_2$
1_1	I_1	—	$\mathrm{I}_1(1)$	$1 \cdot 1$
1_8	E_8	—	$\mathrm{II}_8(1)$	$1 \cdot 1$
1_{12}	D_{12}	[1]	$\mathrm{I}_{12}(1)$	$1 \cdot 1$
1_{14}	E_7^2	[11]	$\mathrm{I}_{14}(1)$	$1 \cdot 2$
1_{15}	A_{15}	[4]	$\mathrm{I}_{15}(1)$	$2 \cdot 1$
$1'_{16}$	D_8^2	[(12)]	$\mathrm{I}_{16}(1)$	$1 \cdot 2$
$1''_{16}$	D_{16}	[1]	$\mathrm{II}_{16}(1)$	$1 \cdot 1$
1_{17}	$A_{11} E_6$	[41]	$\mathrm{I}_{17}(1)$	$2 \cdot 1$
2_1	A_1	—	$\mathrm{II}_1(2)$	$1 \cdot 1$
2_7	E_7	—	$\mathrm{II}_7(2)$	$1 \cdot 1$
2_{11}	$D_{10} A_1$	[11]	$\mathrm{I}_{11}(2)$	$1 \cdot 1$
2_{13}	$E_7 D_6$	[11]	$\mathrm{I}_{13}(2)$	$1 \cdot 1$
2_{14}	$A_{13} 7_1$	[4 1/7]	$\mathrm{I}_{14}(2)$	$2 \cdot 1$
$2'_{15}$	$D_8 D_6 A_1$	[1 1 1,3 2 0]	$\mathrm{I}_{15}(2)$	$1 \cdot 1$
$2''_{15}$	$D_{14} A_1$	[1 1]	$\mathrm{II}_{15}(2)$	$1 \cdot 1$
3_1	3_1	—	$\mathrm{I}_1(3^+)$	$2 \cdot 1$
3_2	A_2	—	$\mathrm{II}_2(3^-)$	$2 \cdot 1$
3_6	E_6	—	$\mathrm{II}_6(3^+)$	$2 \cdot 1$
3_8	$E_7 6_1$	[1 1/2]	$\mathrm{I}_8(3^-)$	$2 \cdot 1$
3_{10}	$D_9 12_1$	[1 1/4]	$\mathrm{I}_{10}(3^+)$	$2 \cdot 1$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
3_{11}	A_{11}	[6]	$\mathrm{I}_{11}(3^-)$	$2 \cdot 1$
3^+_{12}	$E_7 A_5$	[1 3]	$\mathrm{I}_{12}(3^+)$	$2 \cdot 1$
3^-_{12}	$D_{10} A_1 6_1$	[1 1 0, 3 0 ½]	$\mathrm{I}_{12}(3^-)$	$2 \cdot 1$
3^+_{13}	$A_{12} 39_1$	[4 1/13]	$\mathrm{I}_{13}(3^+)$	$2 \cdot 1$
3^-_{13}	$E_6^2 3_1$	[1 1 ⅓]	$\mathrm{I}_{13}(3^-)$	$2 \cdot 2$
4_1	4_1	—	$\mathrm{II}_1(4^+_+)$	$2 \cdot 1$
4_3	A_3	—	$\mathrm{II}_3(4^-_-)$	$2 \cdot 1$
4_4	D_4	—	$\mathrm{II}_4(2^{2-})$	$6 \cdot 1$
4_5	D_5	—	$\mathrm{II}_5(4^-_+)$	$2 \cdot 1$
4_6	D_6	—	$\mathrm{II}_6(2_- 2_-)$	$2 \cdot 1$
4_7	D_7	—	$\mathrm{II}_7(4^+_-)$	$2 \cdot 1$
4_8	D_8	—	$\mathrm{II}_8(2^{2+})$	$2 \cdot 1$
$4'_{+9}$	$D_8 4_1$	[1 ½]	$\mathrm{I}_9(4_+)$	$2 \cdot 1$
$4'_{-9}$	$E_7 A_1 4_1$	[1 1 ½]	$\mathrm{I}_9(4_-)$	$2 \cdot 1$
$4''_9$	D_9	—	$\mathrm{II}_9(4^+_+)$	$2 \cdot 1$
$(2^2)_{10}$	$E_7 A_1^3$	[1 1 1 1]	$\mathrm{I}_{10}(2^2)$	$1 \cdot 6$
$(2 \cdot 2)_{10}$	$D_8 A_1^2$	[1 1 1]	$\mathrm{I}_{10}(2 \cdot 2)$	$1 \cdot 2$
$4''_{10}$	D_{10}	—	$\mathrm{II}_{10}(2_+ 2_+)$	$2 \cdot 1$
$4'_{+11}$	$E_7 A_3 A_1$	[1 2 1]	$\mathrm{I}_{11}(4_+)$	$2 \cdot 1$
$4'_{-11}$	$D_8 A_3$	[1 2]	$\mathrm{I}_{11}(4_-)$	$2 \cdot 1$
$4''_{11}$	D_{11}	—	$\mathrm{II}_{11}(4^-_-)$	$2 \cdot 1$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
5_1	5_1	—	$\mathrm{I}_1(5^+)$	$2 \cdot 1$
5_2	$A_1 10_1$	$[1\ \tfrac{1}{2}]$	$\mathrm{I}_2(5^-)$	$2 \cdot 1$
5_4	A_4	—	$\mathrm{II}_4(5^+)$	$2 \cdot 1$
5_7	$E_6 15_1$	$[1\ \tfrac{1}{3}]$	$\mathrm{I}_7(5^-)$	$2 \cdot 1$
5_8^+	$D_7 20_1$	$[1\ \tfrac{1}{4}]$	$\mathrm{I}_8(5^+)$	$2 \cdot 1$
5_8^-	$E_7 10_1$	$[1\ \tfrac{1}{2}]$	$\mathrm{II}_8(5^-)$	$2 \cdot 1$
5_{10}^+	$E_7 A_2 30_1$	$[1\ 1\ \tfrac{1}{6}]$	$\mathrm{I}_{10}(5^+)$	$2 \cdot 1$
5_{10}^-	$A_9 A_1$	$[5\ 1]$	$\mathrm{I}_{10}(5^-)$	$2 \cdot 1$
5_{11}^+	$A_{10} 55_1$	$[4\ \tfrac{1}{11}]$	$\mathrm{I}_{11}(5^+)$	$2 \cdot 1$
5_{11}^-	$D_9 (3^1 7)_2$	$[1\ \tfrac{1}{4}\ \tfrac{1}{4}]$	$\mathrm{I}_{11}(5^-)$	$2 \cdot 1$
6_1	6_1	—	$\mathrm{II}_1(3^- \cdot 2)$	$2 \cdot 1$
6_5	A_5	—	$\mathrm{II}_5(3^+ \cdot 2)$	$2 \cdot 1$
6_7	$D_6 6_1$	$[1\ \tfrac{1}{2}]$	$\mathrm{I}_7(3^- \cdot 2)$	$2 \cdot 1$
6_9	$D_7 A_1 12_1$	$[1\ 1\ \tfrac{1}{4}]$	$\mathrm{I}_9(3^+ \cdot 2)$	$2 \cdot 1$
6_{10}	$A_9 15_1$	$[4\ \tfrac{1}{5}]$	$\mathrm{I}_{10}(3^- \cdot 2)$	$2 \cdot 1$
7_1	7_1	—	$\mathrm{I}_1(7^+)$	$2 \cdot 1$
7_2	$A_1 14_1$	$[1\ \tfrac{1}{2}]$	$\mathrm{II}_2(7^+)$	$2 \cdot 1$
7_3	$A_2 21_1$	$[1\ \tfrac{1}{3}]$	$\mathrm{I}_3(7^-)$	$2 \cdot 1$
7_6^+	$D_5 28_1$	$[1\ \tfrac{1}{4}]$	$\mathrm{I}_6(7^+)$	$2 \cdot 1$
7_6^-	A_6	—	$\mathrm{II}_6(7^-)$	$2 \cdot 1$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
7_8^+	$E_7\,14_1$	[1 ½]	$\mathrm{I}_8(7^+)$	$2 \cdot 1$
7_8^-	$E_6A_1\,42_1$	[1 1 ⅙]	$\mathrm{I}_8(7^-)$	$2 \cdot 1$
7_9^+	$A_8\,63_1$	[4 ⅑]	$\mathrm{I}_9(7^+)$	$2 \cdot 1$
7_9^-	$E_7(3^15)_2$	[1 ½ ½]	$\mathrm{I}_9(7^-)$	$2 \cdot 1$
$(7^+)'_{10}$	$D_8(4^28)_2$	[1 ½ 0, 2 0 ½]	$\mathrm{I}_{10}(7^+)$	$2 \cdot 1$
$(7^-)'_{10}$	$D_7A_2\,84_1$	[1 1 1/12]	$\mathrm{I}_{10}(7^-)$	$2 \cdot 1$
$7''_{10}$	$D_9\,28_1$	[1 ¼]	$\mathrm{II}_{10}(7^+)$	$2 \cdot 1$
7_{11}^+	$E_6A_4\,105_1$	[1 2 1/15]	$\mathrm{I}_{11}(7^+)$	$2 \cdot 1$
7_{11}^-	$A_9A_1\,35_1$	[3 1 ⅕]	$\mathrm{I}_{11}(7^-)$	$2 \cdot 1$
8_1	8_1	—	$\mathrm{II}_1(8_+^+)$	$2 \cdot 1$
8_2	$(3^13)_2$	—	$\mathrm{I}_2(8_-^-)$	$2^2 \cdot 1$
8_3	$A_1^2 8_1$	[1 1 ½]	$\mathrm{I}_3(8_+^-)$	$2 \cdot 2$
8_4	$A_3\,8_1$	[2 ½]	$\mathrm{I}_4(8_-^+)$	$2^2 \cdot 1$
8_5	$D_4\,8_1$	[2 ½]	$\mathrm{I}_5(8_+^+)$	$2^2 \cdot 1$
8_6	$D_5\,8_1$	[2 ½]	$\mathrm{I}_6(8_-^-)$	$2^2 \cdot 1$
$8'_7$	$D_6\,8_1$	[2 ½]	$\mathrm{I}_7(8_+^-)$	$2^2 \cdot 1$
8_{-7}^+	A_7	—	$\mathrm{II}_7(8_-^+)$	$2 \cdot 1$
8_{-7}^-	$E_6\,24_1$	[1 ⅓]	$\mathrm{II}_7(8_-^-)$	$2 \cdot 1$
8_{-8}^+	$D_7\,8_1$	[2 ½]	$\mathrm{I}_8(8_-^+)$	$2^2 \cdot 1$
8_{+8}^-	$E_6(4^27)_2$	[1 ⅓ ⅓]	$\mathrm{I}_8(8_+^-)$	$2^2 \cdot 1$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
8^{+}_{+8}	$A_7 4_1$	[4 ½]	$\mathrm{I}_8(8^{+}_{+})$	$2^2 \cdot 1$
$(4 \cdot 2)_8$	$D_6 A_1 4_1$	[1 1 ½]	$\mathrm{I}_8(4 \cdot 2)$	$2 \cdot 1$
$(8^{+}_{+})'_9$	$D_8 8_1$	[2 ½]	$\mathrm{I}_9(8^{+}_{+})$	$2^2 \cdot 1$
$(8^{+}_{-})'_9$	$E_6 A_1^2 24_1$	[1 1 1 ⅙]	$\mathrm{I}_9(8^{+}_{-})$	$2 \cdot 2$
$(8^{-}_{-})'_9$	$A_7 A_1^2$	[4 1 1]	$\mathrm{I}_9(8^{-}_{-})$	$2 \cdot 2$
$(2 \cdot 2 \cdot 2)_9$	$D_6 A_1^3$	[1 1 1 1]	$\mathrm{I}_9(2 \cdot 2 \cdot 2)$	$1 \cdot 3!$
$(8^{+})''_9$	$D_8 8_1$	[1 ½]	$\mathrm{II}_9(8^{+}_{+})$	$2 \cdot 1$
$(8^{-})''_9$	$E_7 A_1 8_1$	[1 1 ½]	$\mathrm{II}_9(8^{-}_{+})$	$2 \cdot 1$
9_1	9_1	—	$\mathrm{I}_1(9^{+})$	$2 \cdot 1$
9_2	$A_1 18_1$	[1 ½]	$\mathrm{I}_2(9^{-})$	$2 \cdot 1$
9_4	$A_3 36_1$	[1 ¼]	$\mathrm{I}_4(9^{+})$	$2 \cdot 1$
9_5	$A_4 45_1$	[2 ⅕]	$\mathrm{I}_5(9^{-})$	$2 \cdot 1$
9_6	$A_5 6_1$	[3 ½]	$\mathrm{I}_6(3^{2-})$	$2^2 \cdot 1$
9_7	$A_6 63_1$	[3 ⅐]	$\mathrm{I}_7(9^{+})$	$2 \cdot 1$
10_1	10_1	—	$\mathrm{II}_1(5^{-} \cdot 2)$	$2 \cdot 1$
10_3	$A_2 30_1$	[1 ⅓]	$\mathrm{II}_3(5^{+} \cdot 2)$	$2 \cdot 1$
10_6	$A_5 15_1$	[2 ⅓]	$\mathrm{I}_6(5^{-} \cdot 2)$	$2 \cdot 1$
$10'_7$	$D_5 A_1 20_1$	[1 1 ¼]	$\mathrm{I}_7(5^{+} \cdot 2)$	$2 \cdot 1$
$10''_7$	$D_6 10_1$	[1 ½]	$\mathrm{II}_7(5^{-} \cdot 2)$	$2 \cdot 1$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
11_1	11_1	—	$\mathrm{I}_1(11^+)$	$2 \cdot 1$
11^{+}_{2}	$(3^1 4)_2$	—	$\mathrm{I}_2(11^+)$	$2 \cdot 1$
11^{-}_{2}	$A_1 22_1$	$[1\ \frac{1}{2}]$	$\mathrm{II}_2(11^-)$	$2 \cdot 1$
11_4	$A_2 A_1 66_1$	$[1\ 1\ \frac{1}{6}]$	$\mathrm{I}_4(11^-)$	$2 \cdot 1$
11_5	$A_4 55_1$	$[1\ \frac{1}{5}]$	$\mathrm{I}_5(11^+)$	$2 \cdot 1$
11_6	$D_5 44_1$	$[1\ \frac{1}{4}]$	$\mathrm{II}_6(11^+)$	$2 \cdot 1$
11^{+}_{7}	$E_6 33_1$	$[1\ \frac{1}{3}]$	$\mathrm{I}_7(11^+)$	$2 \cdot 1$
11^{-}_{7}	$A_6 77_1$	$[2\ \frac{1}{7}]$	$\mathrm{I}_7(11^-)$	$2 \cdot 1$
12_1	12_1	—	$\mathrm{II}_1(3^+ \cdot 4^{-}_{-})$	$2 \cdot 1$
12_2	$(4^2 4)_2$	—	$\mathrm{II}_2(3^+ \cdot 2^{2-})$	$12 \cdot 1$
$12'_3$	$A_1 4_1 6_1$	$[1\ \frac{1}{2}\ \frac{1}{2}]$	$\mathrm{I}_3(3^- \cdot 4^-)$	$2^2 \cdot 1$
$12''_3$	$A_1^2 12_1$	$[1\ 1\ \frac{1}{2}]$	$\mathrm{II}_3(3^+ \cdot 4^{+}_{+})$	$2 \cdot 2$
$12'_4$	$A_1^3 6_1$	$[1\ 1\ 1\ \frac{1}{2}]$	$\mathrm{I}_4(3^- \cdot 2^2)$	$2 \cdot 3!$
$12''_4$	$A_3 12_1$	$[2\ \frac{1}{2}]$	$\mathrm{II}_4(3^+ \cdot 2 \cdot 2)$	$2^2 \cdot 1$
$12'_5$	$A_3 A_1 6_1$	$[2\ 1\ \frac{1}{2}]$	$\mathrm{I}_5(3^- \cdot 4_+)$	$2^2 \cdot 1$
$12''_5$	$D_4 12_1$	$[2\ \frac{1}{2}]$	$\mathrm{II}_5(3^+ \cdot 4^{+}_{-})$	$2^2 \cdot 1$
$12'_6$	$D_4 A_1 6_1$	$[2\ 1\ \frac{1}{2}]$	$\mathrm{I}_6(3^- \cdot 2 \cdot 2)$	$2^2 \cdot 1$
$12''_6$	$D_5 12_1$	$[2\ \frac{1}{2}]$	$\mathrm{II}_6(3^+ \cdot 2^{2+})$	$2^2 \cdot 1$
12^{+}_{+7}	$D_5 4_1 12_1$	$[1\ \frac{1}{2}\ \frac{1}{4}]$	$\mathrm{I}_7(3^+ \cdot 4_+)$	$2^2 \cdot 1$
12^{+}_{-7}	$A_5 A_1 4_1$	$[3\ 1\ \frac{1}{2}]$	$\mathrm{I}_7(3^+ \cdot 4_-)$	$2^2 \cdot 1$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
12^{-}_{-7}	$D_5 A_1 6_1$	$[2\ 1\ \tfrac{1}{2}]$	$\mathrm{I}_7(3^- \cdot 4_-)$	$2^2 \cdot 1$
$12''_7$	$D_6 12_1$	$[2\ \tfrac{1}{2}]$	$\mathrm{II}_7(3^+ \cdot 4^+_+)$	$2^2 \cdot 1$
12^{-}_{+8}	$A_7 24_1$	$[2\ \tfrac{1}{4}]$	$\mathrm{I}_8(3^- \cdot 4_+)$	$2 \cdot 1$
12^{-}_{-8}	$E_6 3_1 12_1$	$[1\ \tfrac{1}{3}\ \tfrac{1}{3}]$	$\mathrm{I}_8(3^- \cdot 4_-)$	$2 \cdot 1$
$(3^+ \cdot 2^2)_8$	$A_5 A_1^3$	$[3\ 1\ 1\ 1]$	$\mathrm{I}_8(3^+ \cdot 2^2)$	$2 \cdot 3!$
$(3^- \cdot 2^2)_8$	$D_6 A_1 6_1$	$[2\ 1\ \tfrac{1}{2}]$	$\mathrm{I}_8(3^- \cdot 2^2)$	$2^2 \cdot 1$
$(3 \cdot 2 \cdot 2)'_8$	$D_5 A_1^2 12_1$	$[1\ 1\ 1\ \tfrac{1}{4}]$	$\mathrm{I}_8(3^+ \cdot 2 \cdot 2)$	$2 \cdot 2$
$12''_8$	$D_7 12_1$	$[2\ \tfrac{1}{2}]$	$\mathrm{II}_8(3^+ \cdot 2 \cdot 2)$	$2^2 \cdot 1$
13_1	13_1	—	$\mathrm{I}_1(13^+)$	$2 \cdot 1$
13_2	$A_1 26_1$	$[1\ \tfrac{1}{2}]$	$\mathrm{I}_2(13^-)$	$2 \cdot 1$
13^+_3	$A_2 39_1$	$[1\ \tfrac{1}{3}]$	$\mathrm{I}_3(13^+)$	$2 \cdot 1$
13^-_3	$A_1 (3^1 9)_2$	$[1\ \tfrac{1}{2}\ \tfrac{1}{2}]$	$\mathrm{I}_3(13^-)$	$2 \cdot 1$
13_4	$A_3 52_1$	$[1\ \tfrac{1}{4}]$	$\mathrm{II}_4(13^+)$	$2 \cdot 1$
13^+_6	$A_4 A_1 130_1$	$[2\ 1\ \tfrac{1}{10}]$	$\mathrm{I}_6(13^+)$	$2 \cdot 1$
13^-_6	$A_5 78_1$	$[1\ \tfrac{1}{6}]$	$\mathrm{I}_6(13^-)$	$2 \cdot 1$
13_7	$D_5 (7^2 8)_2$	$[1\ \tfrac{1}{2}\ \tfrac{1}{4}]$	$\mathrm{I}_7(13^-)$	$2 \cdot 1$
14_1	14_1	—	$\mathrm{II}_1(7^+ \cdot 2)$	$2 \cdot 1$
14_2	$(3^1 5)_2$	—	$\mathrm{I}_2(7^- \cdot 2)$	$2 \cdot 1$
$14'_5$	$A_3 A_1 28_1$	$[1\ 1\ \tfrac{1}{4}]$	$\mathrm{I}_5(7^+ \cdot 2)$	$2 \cdot 1$
$14''_5$	$A_4 70_1$	$[2\ \tfrac{1}{5}]$	$\mathrm{II}_5(7^- \cdot 2)$	$2 \cdot 1$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
$(14^+)'_7$	$D_6 14_1$	[1 ½]	$\mathrm{I}_7(7^+ \cdot 2)$	$2 \cdot 1$
$(14^-)'_7$	$A_5 A_1 42_1$	[2 1 ⅙]	$\mathrm{I}_7(7^- \cdot 2)$	$2 \cdot 1$
$14''_7$	$E_6 42_1$	[1 ⅓]	$\mathrm{II}_7(7^- \cdot 2)$	$2 \cdot 1$
14^+_8	$A_6 7_1 14_1$	[2 3/7 1/7]	$\mathrm{I}_8(7^+ \cdot 2)$	$2 \cdot 1$
14^-_8	$D_6(3^1 5)_2$	[1 ½ ½]	$\mathrm{I}_8(7^- \cdot 2)$	$2 \cdot 1$
15_1	15_1	—	$\mathrm{I}_1(5^- \cdot 3^-)$	$2 \cdot 1$
15^{++}_2	$A_1 30_1$	[1 ½]	$\mathrm{II}_2(5^+ \cdot 3^+)$	$2 \cdot 1$
15^{--}_2	$(4^1 4)_2$	—	$\mathrm{II}_2(5^- \cdot 3^-)$	$2^2 \cdot 1$
15_4	$A_2(6^3 9)_2$	[1 ⅓ ⅓]	$\mathrm{I}_4(5^+ \cdot 3^-)$	$2^2 \cdot 1$
15_5	$A_2^2 15_1$	[1 1 ⅓]	$\mathrm{I}_5(5^- \cdot 3^+)$	$2 \cdot 2$
$15''_6$	$A_5 10_1$	[3 ½]	$\mathrm{II}_6(5^- \cdot 3^+)$	$2^2 \cdot 1$
15^{++}_6	$D_4(8^2 8)_2$	[1 ½ 0, 3 0 ½]	$\mathrm{I}_6(5^+ \cdot 3^+)$	$2^2 \cdot 1$
15^{--}_6	$D_5 60_1$	[1 ¼]	$\mathrm{I}_6(5^- \cdot 3^-)$	$2 \cdot 1$
15^{+-}_7	$A_6 105_1$	[1 1/7]	$\mathrm{I}_7(5^+ \cdot 3^-)$	$2 \cdot 1$
15^{--}_7	$A_5 6_1 15_1$	[1 ½ ⅓]	$\mathrm{I}_7(5^- \cdot 3^-)$	$2^2 \cdot 1$
16_1	16_1	—	$\mathrm{II}_1(16^+_+)$	$2 \cdot 1$
16_2	$(4^2 5)_2$	—	$\mathrm{I}_2(16^-_+)$	$2^2 \cdot 1$
$(4^2)_3$	$(3^{-1}\ 3^{-1}\ 3^{-1})_3$	—	$\mathrm{I}_3(4^{2-})$	$48 \cdot 1$
16^+_{+3}	$A_1^2 16_1$	[1 1 ½]	$\mathrm{I}_3(16^+_+)$	$2 \cdot 2$
$16''_3$	$A_2 48_1$	[1 ⅓]	$\mathrm{II}_3(16^-_-)$	$2 \cdot 1$
16^+_{-4}	$A_2(4^2 13)_2$	[1 ⅓ ⅓]	$\mathrm{I}_4(16^+_-)$	$2^2 \cdot 1$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
16^{-}_{+4}	$A_3 16_1$	[2 ½]	$I_4(16^{-}_{+})$	$2^2 \cdot 1$
$(4 \cdot 4)_4$	$A_1^2 4_1^2$	[1 1 ½ ½]	$I_4(4_{-} \cdot 4_{-})$	$2^2 \cdot 2^2$
16^{+}_{+5}	$D_4 16_1$	[2 ½]	$I_5(16^{+}_{+})$	$2^2 \cdot 1$
16^{-}_{+5}	$A_4 80_1$	[1 ⅕]	$II_5(16^{-}_{+})$	$2 \cdot 1$
16^{-}_{-5}	$A_2 A_1^2 48_1$	[1 1 1 ⅙]	$I_5(16^{-}_{-})$	$2 \cdot 2$
$(4^2)_5$	$A_3 4_1^2$	[2 ½ ½]	$I_5(4^{2+})$	$2^3 \cdot 2$
$(4 \cdot 2^2)_5$	$A_1^4 4_1$	[1 1 1 1 ½]	$I_5(4_{-} \cdot 2^2)$	$2 \cdot 4!$
16^{+}_{+6}	$A_4(4^2 21)_2$	[2 ⅖ ⅕]	$I_6(16^{+}_{+})$	$2^2 \cdot 1$
16^{+}_{-6}	$A_3 A_2 48_1$	[2 1 ⅙]	$I_6(16^{+}_{-})$	$2^2 \cdot 1$
16^{-}_{+6}	$D_5 16_1$	[2 ½]	$I_6(16^{-}_{+})$	$2^2 \cdot 1$
$16''_6$	$A_5 24_1$	[2 ⅓]	$II_6(8_{-} \cdot 2)$	$2 \cdot 1$
$(4_{+} \cdot 4_{+})_6$	$D_4 4_1^2$	[2 ½ ½]	$I_6(4_{+} \cdot 4_{+})$	$2^3 \cdot 2$
$(4_{+} \cdot 4_{-})_6$	$A_3 A_1^2 4_1$	[2 1 1 ½]	$I_6(4_{+} \cdot 4_{-})$	$2^2 \cdot 2$
$(2^4)_6$	A_1^6	[1 1 1 1 1 1]	$I_6(2^4)$	$1 \cdot 6!$
16^{+}_{+7}	$D_6 16_1$	[2 ½]	$I_7(16^{+}_{+})$	$2^2 \cdot 1$
16^{+}_{-7}	$A_6 112_1$	[3 ⅐]	$II_7(16^{+}_{-})$	$2 \cdot 1$
16^{-}_{+7}	$A_4 A_1^2 80_1$	[2 1 1 ⅒]	$I_7(16^{-}_{+})$	$2 \cdot 2$
16^{-}_{-7}	$D_4 A_2 48_1$	[2 1 ⅙]	$I_7(16^{-}_{-})$	$2^2 \cdot 1$
$(8 \cdot 2)_7$	$A_5(4^2 7)_2$	[2 ⅓ ⅓]	$I_7(8_{+} \cdot 2)$	$2^2 \cdot 1$
$(4^{2+})_7$	$A_3^2 4_1$	[2 2 ½]	$I_7(4^{2+})$	$2^3 \cdot 2$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
$(4^{2-})_7$	$D_5 4_1^2$	[2 ½ ½]	$\mathrm{I}_7(4^{2-})$	$2^3 \cdot 2$
$(4 \cdot 2^2)_7$	$A_3 A_1^4$	[2 1 1 1 1]	$\mathrm{I}_7(4_+ \cdot 2^2)$	$2 \cdot 4!$
$(4 \cdot 2 \cdot 2)_7$	$D_4 A_1^2 4_1$	[2 1 1 ½]	$\mathrm{I}_7(4 \cdot 2 \cdot 2)$	$2^2 \cdot 2$
17_1	17_1	—	$\mathrm{I}_1(17^+)$	$2 \cdot 1$
17_2^+	$A_1 34_1$	[1 ½]	$\mathrm{I}_2(17^+)$	$2 \cdot 1$
17_2^-	$(3^1 6)_2$	—	$\mathrm{I}_2(17^-)$	$2 \cdot 1$
17_3	$A_1(5^1 7)_2$	[1 ½ ½]	$\mathrm{I}_3(17^-)$	$2 \cdot 1$
17_4^+	$A_3 68_1$	[1 ¼]	$\mathrm{I}_4(17^+)$	$2 \cdot 1$
17_4^-	$A_2 A_1 102_1$	[1 1 ⅙]	$\mathrm{II}_4(17^-)$	$2 \cdot 1$
17_5	$A_3(8^2 9)_2$	[1 ¼ ½]	$\mathrm{I}_5(17^+)$	$2 \cdot 1$
17_6	$A_4 A_1 170_1$	[1 1 1/10]	$\mathrm{I}_6(17^-)$	$2 \cdot 1$
18_1	18_1	—	$\mathrm{II}_1(9^- \cdot 2)$	$2 \cdot 1$
18_3	$A_1(4^2 10)_2$	[1 0 ½]	$\mathrm{I}_3(9^+ \cdot 2)$	$2 \cdot 1$
18_4	$A_2 3_1 18_1$	[1 ⅓ ⅓]	$\mathrm{I}_4(9^- \cdot 2)$	$2 \cdot 1$
18_5	$A_3 6_1 12_1$	[1 ½ ¼]	$\mathrm{I}_5(3^{2-} \cdot 2)$	$2^2 \cdot 1$
18_6	$A_4(7^1 13)_2$	[2 ⅖ ⅕]	$\mathrm{I}_6(9^+ \cdot 2)$	$2 \cdot 1$
19_1	19_1	—	$\mathrm{I}_1(19^+)$	$2 \cdot 1$
19_2^+	$(4^1 5)_2$	—	$\mathrm{I}_2(19^+)$	$2 \cdot 1$
19_2^-	$A_1 38_1$	[1 ½]	$\mathrm{II}_2(19^-)$	$2 \cdot 1$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
19_3^+	$A_1(3^1 13)_2$	$[1\ \frac{1}{2}\ \frac{1}{2}]$	$\mathrm{I}_3(19^+)$	$2 \cdot 1$
19_3^-	$A_2 57_1$	$[1\ \frac{1}{3}]$	$\mathrm{I}_3(19^-)$	$2 \cdot 1$
19_4	$A_1^2(8^2 10)_2$	$[1\ 1\ \frac{1}{2}\ 0,\ 1\ 0\ 0\ \frac{1}{2}]$	$\mathrm{I}_4(19^-)$	$2 \cdot 1$
19_5	$A_4 95_1$	$[2\ \frac{1}{5}]$	$\mathrm{I}_5(19^+)$	$2 \cdot 1$
20_1	20_1	—	$\mathrm{I}_1(5^+ \cdot 4_+^-)$	$2 \cdot 1$
$20_2'$	$(3^1 7)_2$	—	$\mathrm{I}_2(5^- \cdot 4_-^-)$	$2 \cdot 1$
$20_2''$	$(4^2 6)_2$	—	$\mathrm{II}_2(5^+ \cdot 2_- \cdot 2_-)$	$2^2 \cdot 1$
$20_3'$	$(3^1 3^1 3^1)_3$	—	$\mathrm{I}_3(5^- \cdot 2^2)$	$12 \cdot 1$
20_3^+	$A_1^2 20_1$	$[1\ 1\ \frac{1}{2}]$	$\mathrm{II}_3(5^+ \cdot 4_-^+)$	$2 \cdot 2$
20_3^-	$A_1 4_1 10_1$	$[1\ \frac{1}{2}\ \frac{1}{2}]$	$\mathrm{II}_3(5^- \cdot 4_-^-)$	$2^2 \cdot 1$
$20_4'$	$A_1^2(3^1 7)_2$	$[1\ 1\ \frac{1}{2}\ \frac{1}{2}]$	$\mathrm{I}_4(5^- \cdot 4_+^-)$	$2 \cdot 2$
$(5^+ \cdot 2^2)_4$	$A_3 20_1$	$[2\ \frac{1}{2}]$	$\mathrm{II}_4(5^+ \cdot 2^{2+})$	$2^2 \cdot 1$
$(5^- \cdot 2^2)_4$	$A_1^3 10_1$	$[1\ 1\ 1\ \frac{1}{2}]$	$\mathrm{II}_4(5^- \cdot 2^{2-})$	$2 \cdot 3!$
$(20_+)_5'$	$A_3 4_1 20_1$	$[1\ \frac{1}{2}\ \frac{1}{4}]$	$\mathrm{I}_5(5^+ \cdot 4_+)$	$2^2 \cdot 1$
$(20^+)_5''$	$D_4 20_1$	$[2\ \frac{1}{2}]$	$\mathrm{II}_5(5^+ \cdot 4_+^+)$	$2^2 \cdot 1$
$(20^-)_5''$	$A_3 A_1 10_1$	$[2\ 1\ \frac{1}{2}]$	$\mathrm{II}_5(5^- \cdot 4_+^-)$	$2^2 \cdot 1$
$(20_-)_5'$	$A_2 A_1 4_1 30_1$	$[1\ 1\ \frac{1}{2}\ \frac{1}{6}]$	$\mathrm{I}_5(5^+ \cdot 4_-)$	$2^2 \cdot 1$
$(5 \cdot 2 \cdot 2)_5$	$A_3(3^1 7)_2$	$[2\ \frac{1}{2}\ \frac{1}{2}]$	$\mathrm{I}_5(5^- \cdot 2_- \cdot 2_-)$	$2^2 \cdot 1$
$(5 \cdot 4)_6'$	$D_4(3^1 7)_2$	$[2\ \frac{1}{2}\ \frac{1}{2}]$	$\mathrm{I}_6(5^- \cdot 4_-)$	$2^2 \cdot 1$
$(5 \cdot 2^2)_6'$	$A_2 A_1^3 30_1$	$[1\ 1\ 1\ 1\ \frac{1}{6}]$	$\mathrm{I}_6(5^+ \cdot 2^2)$	$2 \cdot 3!$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
$(5 \cdot 2 \cdot 2)'_6$	$A_3 A_1^2 20_1$	$[1\ 1\ 1\ {}^1/_4]$	$\mathrm{I}_6(5^+ \cdot 2 \cdot 2)$	$2 \cdot 2$
$(5^+ \cdot 2 \cdot 2)''_6$	$D_5 20_1$	$[2\ {}^1/_2]$	$\mathrm{II}_6(5^+ \cdot 2_+ \cdot 2_+)$	$2^2 \cdot 1$
$(5^- \cdot 2 \cdot 2)''_6$	$D_4 A_1 10_1$	$[2\ 1\ {}^1/_2]$	$\mathrm{II}_6(5^- \cdot 2_- \cdot 2_-)$	$2^2 \cdot 1$
21_1	21_1	—	$\mathrm{I}_1(7^- \cdot 3^+)$	$2 \cdot 1$
21_2^{+-}	$(5^2 5)_2$	—	$\mathrm{I}_2(7^+ \cdot 3^-)$	$2^2 \cdot 1$
21_2^{--}	$A_1 42_1$	$[1\ {}^1/_2]$	$\mathrm{I}_2(7^- \cdot 3^-)$	$2 \cdot 1$
21_3	$(3^1 3^1 3^0)_3$	—	$\mathrm{I}_3(7^- \cdot 3^-)$	$2^2 \cdot 1$
$21'_4$	$A_1^2(10^4 10)_2$	$[1\ 0\ {}^1/_2\ 0,\ 0\ 1\ 0\ {}^1/_2]$	$\mathrm{I}_4(7^+ \cdot 3^+)$	$2 \cdot 2$
$21''_4$	$A_3 84_1$	$[1\ {}^1/_4]$	$\mathrm{II}_4(7^- \cdot 3^+)$	$2 \cdot 1$
21_5	$A_4 105_1$	$[1\ {}^1/_5]$	$\mathrm{I}_5(7^+ \cdot 3^-)$	$2 \cdot 1$
21_6^{++}	$A_5 14_1$	$[3\ {}^1/_2]$	$\mathrm{I}_6(7^+ \cdot 3^+)$	$2^2 \cdot 1$
21_6^{+-}	$A_3 A_1 6_1 28_1$	$[1\ 1\ 0\ {}^1/_4,\ 2\ 1\ {}^1/_2\ 0]$	$\mathrm{I}_6(7^+ \cdot 3^-)$	$2^2 \cdot 1$
21_6^{-+}	$A_2^2 A_1 42_1$	$[1\ 1\ 1\ {}^1/_6]$	$\mathrm{I}_6(7^- \cdot 3^+)$	$2 \cdot 2$
21_6^{--}	$A_4(6^3 19)_2$	$[1\ {}^2/_5\ {}^1/_5]$	$\mathrm{I}_6(7^- \cdot 3^-)$	$2^2 \cdot 1$
21_7^{++}	$A_5(9^3 15)_2$	$[1\ {}^1/_2\ {}^1/_6]$	$\mathrm{I}_7(7^+ \cdot 3^+)$	$2 \cdot 1$
21_7^{-+}	$A_5(3^1 5)_2$	$[3\ {}^1/_2\ {}^1/_2]$	$\mathrm{I}_7(7^- \cdot 3^+)$	$2^2 \cdot 1$
21_7^{--}	$D_5(8^2 11)_2$	$[1\ {}^1/_4\ {}^1/_2]$	$\mathrm{I}_7(7^- \cdot 3^-)$	$2 \cdot 1$
22_1	22_1	—	$\mathrm{II}_1(11^- \cdot 2)$	$2 \cdot 1$
$22'_3$	$A_1(6^2 8)_2$	$[1\ {}^1/_2\ {}^1/_2]$	$\mathrm{I}_3(11^- \cdot 2)$	$2 \cdot 1$
$22''_3$	$A_2 66_1$	$[1\ {}^1/_3]$	$\mathrm{II}_3(11^- \cdot 2)$	$2 \cdot 1$

Table 1 (cont'd)

Name	*Shape*	*Glue*	*Genus*	$g_1 \cdot g_2$
22_4	$A_2(7^2 10)_2$	$[1\ \frac{1}{3}\ \frac{1}{3}]$	$\mathrm{I}_4(11^+ \cdot 2)$	$2 \cdot 1$
22_5	$A_3 A_1 44_1$	$[1\ 1\ \frac{1}{4}]$	$\mathrm{II}_5(11^+ \cdot 2)$	$2 \cdot 1$
22_6^+	$A_5 33_1$	$[2\ \frac{1}{3}]$	$\mathrm{I}_6(11^+ \cdot 2)$	$2 \cdot 1$
22_6^-	$A_4(3^1 37)_2$	$[1\ \frac{1}{5}\ \frac{2}{5}]$	$\mathrm{I}_6(11^- \cdot 2)$	$2 \cdot 1$
23_1	23_1	—	$\mathrm{I}_1(23^+)$	$2 \cdot 1$

Table 2.

d_n	*Lattices*
1_{18}	$A_{17}A_1[3\ 1] \sim D_{10}E_7A_1[1\ 1\ 0,\ 3\ 0\ 1] \sim A_9^2[1\ 3] \sim D_6^3[(2\ 1\ 1)]$
2_{16}	$A_{11}A_5[2\ 2] \sim A_9E_6 15_1[2\ 1\ \tfrac{1}{15}]$
3_{14}	$D_{13}12_1[1\ \tfrac14]$, $A_{11}A_3[31] \sim E_7D_6 6_1\ [1\ 1\ 0,\ 0\ 3\ \tfrac12]$, $D_8D_5 12_1[1\ 2\ 0,\ 3\ 3\ \tfrac14]$
4_{12}	D_{12}, $A_{11}3_1[4\ \tfrac13]$, $A_{11}12_1[2\ \tfrac13]$, $E_7D_4A_1\ [1\ 1\ 1] \sim D_6^2[1\ 1]$, $D_8D_4[1\ 1]$
5_{12}	$D_{11}20_1[1\ \tfrac14]$, $D_{10}A_1 10_1[1\ 1\ 0,\ 3\ 0\ \tfrac12] \sim E_6D_5 60_1[1\ 1\ \tfrac1{12}]$, $D_8A_3 20_1[3\ 2\ 0,\ 1\ 1\ \tfrac14]$
6_{11}	$D_{10}6_1[1\ \tfrac12]$, $D_8A_1^2 6_1[1\ 1\ 1\ 0,\ 2\ 0\ 1\ \tfrac12]$, $E_7A_3 12_1[1\ 1\ \tfrac14] \sim D_6A_5[1\ 3]$
7_{12}	$A_{11}21_1[2\ \tfrac13] \sim D_9A_1(6^2 10)_2[0\ 1\ 0\ \tfrac12,\ 1\ 1\ \tfrac14\ \tfrac14] \sim E_7A_4 70_1[1\ 1\ \tfrac1{10}] \sim$ $D_6A_5 42_1[3\ 3\ 0,\ 1\ 2\ \tfrac16]$, $D_{10}A_1 14_1[1\ 1\ 0,\ 3\ 0\ \tfrac12] \sim A_{10}A_1 154_1[3\ 1\ \tfrac1{22}] \sim$ $A_9A_2 210_1[3\ 1\ \tfrac1{30}] \sim E_7A_3A_1 28_1[1\ 2\ 1\ 0,\ 0\ 1\ 1\ \tfrac14]$
8_{10}	$D_9 8_1[2\ \tfrac12] \sim D_8(3^1 3)_2[1\ \tfrac12\ \tfrac12]$, $E_7A_1(3^1 3)_2[1\ 1\ \tfrac12\ \tfrac12]$, $A_7A_3[4\ 2]$, $E_6A_3 24_1\ [1\ 2\ \tfrac16]$, $D_6A_3A_1[1\ 2\ 1]$
9_8	$A_8 \sim E_7 18_1[1\ \tfrac12]$, $D_7 36_1[1\ \tfrac14]$, $A_7 72_1[3\ \tfrac18]$, $D_6 6_1^2[1\ \tfrac12\ 0,\ 3\ 0\ \tfrac12]$
10_9	A_9, $E_7(4^2 6)_2[1\ 0\ \tfrac12] \sim D_6A_2 30_1[1\ 1\ \tfrac16]$, $A_7A_1 40_1[3\ 1\ \tfrac18]$
11_8	$E_7 22_1[1\ \tfrac12] \sim D_6(6^2 8)_2[1\ \tfrac12\ 0,\ 2\ 0\ \tfrac12]$, $A_6A_1 154_1[3\ 1\ \tfrac1{14}]$
12_9	$D_8 12_1[1\ \tfrac12]$, $D_8 12_1[2\ \tfrac12]$, $E_7A_1 12_1\ [1\ 1\ \tfrac12] \sim A_5A_3A_1[3\ 2\ 1]$, $E_7 4_1 6_1[1\ \tfrac12\ \tfrac12]$, $D_7A_1 6_1[2\ 1\ \tfrac12]$, $A_7(4^2 7)_2[2\ \tfrac14\ \tfrac12]$, $E_6 3_1(4^2 4)_2\ [1\ \tfrac13\ \tfrac13\ \tfrac13]$, $D_6A_1 4_1 6_1[1\ 1\ \tfrac12\ 0,\ 2\ 1\ 0\ \tfrac12]$, $D_6A_1 4_1 6_1[1\ 1\ \tfrac12\ 0,\ 2\ 1\ \tfrac12\ \tfrac12]$, $D_5A_3 12_1[1\ 2\ \tfrac14]$
13_8	$E_7 26_1[1\ \tfrac12] \sim E_6A_1 78_1\ [1\ 1\ \tfrac16]$, $D_7 52_1[1\ \tfrac14] \sim D_5A_2 156_1\ [1\ 1\ \tfrac1{12}]$, $E_6(5^1 8)_2[1\ \tfrac13\ \tfrac13]$
14_9	$D_7A_1 28_1[1\ 1\ \tfrac14]$, $D_7(6^2 10)_2[1\ \tfrac14\ \tfrac14] \sim D_5A_2A_1 84_1[1\ 1\ 1\ \tfrac1{12}]$, $D_6A_1(4^2 8)_2[1\ 1\ \tfrac12\ 0,\ 2\ 0\ 0\ \tfrac12]$

Table 2 (cont'd)

d_n	*Lattices*
15_8	$E_7 30_1[1\ \tfrac12] \sim A_5A_2 30_1[3\ 1\ \tfrac16]$, $D_6 6_1 10_1[1\ \tfrac12\ 0,\ 3\ 0\ \tfrac12]$, $A_6A_1 210_1[2\ 1\ \tfrac1{14}]$, $D_5A_1 6_1 20_1[1\ 1\ 0\ \tfrac14,\ 2\ 1\ \tfrac12\ 0]$
16_8	$D_7 16_1[2\ \tfrac12]$, $A_7 8_1[4\ \tfrac12]$, $E_6(8^4 8)_2[1\ \tfrac13\ \tfrac13]$, $D_6A_1 8_1[1\ 1\ \tfrac12]$, $D_6 4_1^2[2\ \tfrac12\ \tfrac12] \sim D_5A_1^2 4_1[2\ 1\ 1\ \tfrac12]$, $A_6(4^2 29)_2[2\ \tfrac37\ \tfrac17]$, $D_5A_2 48_1[2\ 1\ \tfrac16]$, $A_5A_1^2 24_1[2\ 1\ 1\ \tfrac16]$, $D_4A_3 4_1[2\ 2\ \tfrac12]$, $D_4A_1^4[2\ 1\ 1\ 1\ 1]$, $A_4A_3 80_1[2\ 2\ \tfrac1{10}]$, $A_3^2A_1^2[2\ 2\ 1\ 1]$
17_7	$E_6 51_1[1\ \tfrac13] \sim D_5(3^1 23)_2[1\ \tfrac14\ \tfrac14]$, $A_4A_2 255_1[1\ 1\ \tfrac1{15}]$
18_7	$D_6 18_1[1\ \tfrac12] \sim A_6 126_1[2\ \tfrac17]$, $D_5A_1 36_1[1\ 1\ \tfrac14]$, $A_5 6_1 18_1[1\ \tfrac16\ \tfrac13]$, $D_4A_1 6_1^2[1\ 1\ \tfrac12\ 0,\ 3\ 1\ 0\ \tfrac12]$
19_6	$D_5 76_1[1\ \tfrac14] \sim A_5 114_1[1\ \tfrac16]$, $A_4(9^2 11)_2[1\ \tfrac25\ \tfrac15]$, $A_3A_2 228_1[1\ 1\ \tfrac1{12}]$
20_7	$E_6 60_1[1\ \tfrac13] \sim D_5A_1 10_1[2\ 1\ \tfrac12]$, $D_6 20_1[2\ \tfrac12]$, $D_5(3^1 7)_2[2\ \tfrac12\ \tfrac12]$, $D_5(8^4 12)_2[1\ \tfrac12\ \tfrac14]$, $A_5A_1 60_1[1\ 1\ \tfrac16]$, $A_3^2 20_1[2\ 1\ \tfrac14]$, $A_3A_2A_1 30_1[2\ 1\ 1\ \tfrac16]$
21_8	$D_7 84_1[1\ \tfrac14] \sim D_4A_2(12^6 24)_2[1\ 1\ \tfrac16\ \tfrac16,\ 2\ 0\ 0\ \tfrac12]$, $E_6 3_1 21_1[1\ \tfrac13\ \tfrac13] \sim A_5A_1 6_1 42_1[1\ 1\ \tfrac12\ \tfrac16,\ 0\ 1\ 0\ \tfrac12]$, $E_6 6_1 42_1[1\ \tfrac12\ \tfrac16] \sim E_7 42_1[1\ \tfrac12]$, $D_6 6_1 14_1[1\ \tfrac12\ 0,\ 3\ 0\ \tfrac12] \sim A_4A_3 420_1[2\ 1\ \tfrac1{20}]$, $D_6(10^4 10)_2[1\ \tfrac12\ 0,\ 3\ 0\ \tfrac12]$, $D_5(5^1 4^1 5^{-1})_3[1\ \tfrac14\ \tfrac12\ \tfrac34]$
22_7	$D_6 22_1[1\ \tfrac12] \sim D_4A_1(6^2 8)_2[1\ 1\ \tfrac12\ 0,\ 2\ 0\ 0\ \tfrac12]$, $A_6 154_1[1\ \tfrac17]$, $A_4A_1(14^2 16)_2[1\ 1\ \tfrac1{10}\ \tfrac3{10}]$
23_2	$A_1 46_1[1\ \tfrac12] \sim (4^1 6)_2$, $(3^1 8)_2$

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